“Non-linearities and upscaling in porous media“

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Philip’s redistribution problem revisited: the role of fluid-fluid interfacial areas

Preprint 2008/3
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May 21, 2008

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Abstract
Recent advances in multi-phase flow theory have shown that the flow of several phases in a porous medium is highly influenced by the interfaces separating these phases. First modeling studies based on this new theory have been performed on a pore scale, as well as on a volume-averaged macro scale using balance equations and constitutive relations that take the role and presence of interfaces into account. However, neither experimental data nor analytical solutions are available on the macro scale so far, although their knowledge is essential for the verification of the new models.

In this paper, we derive a semi-analytical solution for the redistribution of two fluid phases in a horizontal one-dimensional and homogeneous porous medium. We start with the macro-scale model including interfacial area. Next, we construct a semi-analytical solution for this problem by using a similarity transformation. We then compare results obtained from a numerical macro-scale model to this semi-analytical solution used as the reference solution.

1 Motivation
Many environmental, biological, and industrial systems involve flow and transport processes in porous media. Applications range from the movement of contaminants in the subsurface and their remediation, as well as medical applications like brain and liver cancer treatment, to processes occurring during paper manufacturing and in fuel cells (see e.g. [1, 14, 16, 23]).

To define a model for such porous media flow and transport processes, a set of balance equations and constitutive relations is needed. Most modelers use the

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standard approach to describe these systems: the mass balance equation and Darcy's law for each fluid phase, as well as functional relationships describing the dependency of the capillary pressure and of the relative permeability on the saturation [14]. However, it has been shown that this standard approach is deficient and does not always describe the system in a physically correct way. In particular, it does not account for the role and presence of interfaces.

In the last decades, an alternative theory of multi-phase flow and transport in porous media has been developed [9, 10, 11, 12]. This is based on thermodynamic principles and thus, more physically-based than the standard model. The new approach not only includes balance equations for the bulk phases, but also for interfaces. Furthermore, the capillary pressure is not only a function of saturation, but also depends on the interfacial area.

In the standard model, the relationship between capillary pressure and saturation is hysteretic, implying that the capillary pressure is not a single-valued function of saturation, but also depends on the history and on whether the wetting-phase saturation is decreasing (drainage) or increasing (imbibition). Experimental work (see e.g. [17]) has shown the existence of two main (bounding) curves in the saturation - capillary pressure plane. These are the outer curves in Figure 1. The upper curve corresponds to drainage starting at fully wetting phase saturation, whereas the lower curve is obtained for imbibition when starting at fully non-wetting phase saturation. Whenever drainage or imbibition are started at partial wetting phase saturation, the capillary pressure evolves along a secondary drainage or imbibition curve that is included in the area between the bounding ones. Figure 1 displays the primary drainage, main drainage, main imbibition and scanning curves based on experiments by Morrow and Harris (1965). Since the secondary curves have to be defined for every starting point in the saturation-capillary pressure plane, it is quite difficult to construct reliable mathematical models in the standard framework.

The new model includes a capillary pressure–interfacial area–saturation surface. This approach includes different capillary pressure - saturation curves in a natural manner and is based on pore scale modeling studies that have been done up to now for investigating the relationship between the capillary pressure, the interfacial area and the saturation [13, 15, 22]. The parameters involved in the macro-scale balance equations based on the new theory, such as the interfacial velocity, are determined in [20].

As revealed by pore-scale network models, there are minimal differences between the capillary pressure–interfacial area–saturation surfaces that are obtained for imbibition, respectively drainage processes. In this sense we mention the experimental work in [4]. Based on these results, here we disregard the differences between the drainage and imbibition surfaces and assume a unique capillary pressure–interfacial area–saturation surface.

Only a few modeling studies accounting for the interfacial area model were done at the macro scale. For example, in a simple bundle-of-capillary-tubes model presented in [19], the relationship between the average interface velocity and the velocity of the phases was studied. In a macro-scale model developed in [18], a capillary pressure–interfacial area–saturation surface obtained from a
pore-scale model was used. Based on physical reasoning, a formula for the production rate term of the interfacial area was constructed. The model proposed in [18] captures hysteretic effects without specifying explicit hysteretic capillary pressure–saturation functions, but only by introducing a capillary pressure–interfacial area–saturation surface.

Acknowledging the need for macro-scale models accounting for interfaces, it is of crucial interest to find good parameter estimates and verification possibilities for the resulting models. This can be done by comparison either with experimental data, or with analytical and semi-analytical solutions. In the absence of any experimental data on dynamic evolution of interfaces, we focus here on the derivation of a semi-analytical solution for the interfacial area model.

For this purpose, we consider the redistribution of two fluid phases in a one-dimensional horizontal porous medium that is homogeneous and infinitely long. This medium is divided into two parts, with the initial wetting-phase saturation being high in one part and low in the other part, see Figure 2. As redistribution takes place, drainage will occur in the high-saturation part of the domain and imbibition will occur on the other side. The two subdomains are separated by an interface where the saturation becomes discontinuous. Philip [21] obtained an analytical solution for a similar problem. In the framework described above, he considered the case of unsaturated flow using the standard model with two clearly defined capillary pressure–saturation curves (one for the drainage side and one for the imbibition side). Assuming the continuity of the flux, as well as of the capillary pressure, Philip defined a similarity solution for the resulting
Figure 2: Setup of the redistribution problem.

model.

An analogous construction was proposed in [6], where the redistribution problem is studied in the context of an extended capillary pressure condition. The procedure given there is based on the analysis carried out in [2, 3, 7], where the existence and uniqueness of similarity solutions for the nonlinear (possibly degenerate) diffusion equation, as well as monotonicity properties for the associated fluxes, are proven. In the same context we also mention the recent paper [8]. The semi-analytic solutions given there are solving the standard two-phase model including advection terms.

In this work we apply the procedures in the above cited papers to construct similarity solutions for the redistribution problem that includes the interfacial area. This provides a verification tool for numerical models based on the new theory. We proceed as follows: in Section 2, we introduce the system of macro-scale balance equations and constitutive relations that are based on the interfacial area theory, and describe the redistribution problem. In Section 3 we derive the semi-analytical solution to the two-phase redistribution problem, followed by a specific example in Section 4. Finally, Section 5 provides a summary and an outlook on the future research.

2 The redistribution problem

In this section, we derive a simplified model that is based on the interfacial area theory presented in [18].

2.1 The model based on the interfacial area theory

General balance equations for two-phase flow are derived in [10], based on thermodynamic principles. The resulting models are extremely complex, requiring an enormous computational power. Furthermore, several of the involved constitutive relationships have not been investigated so far. Therefore, for deriving a simple extension of the standard two-phase flow model, the following simplifying assumptions are made in [18]:

\[ S_r p_r \quad \text{x < 0} \]

\[ S_r p_r \quad \text{x > 0} \]
• There is no mass transfer between phases.
• The considered physical system is isothermal.
• The porosity does not change in time.
• The phases are incompressible.
• The interfacial mass density is constant.
• The only driving force for the flow of phases is the hydraulic head gradient; further, the driving force for the interfaces is the gradient of specific interfacial area. The standard Darcy law is assumed to hold.
• The cross-coupling terms due to the momentum transfer among phases and interfaces are negligible.
• The porous medium is assumed perfectly wettable to the wetting phase; therefore, the role of solid–fluid interfaces is neglected.

In addition we disregard the gravity, as we are interested in a horizontal physical system. Further we neglect the internal sinks and sources.

These assumptions result in a system of three partial differential equations, one for each of the two phases and one for the interface between the two fluid phases. Additionally, the wetting-phase saturation and the non-wetting phase saturation have to sum up to unity. We also assume quasi-static conditions, so the capillary pressure is the difference between the phase pressures, whereas a functional relationship between the capillary pressure, the specific interfacial area and the wetting-phase saturation has to be provided. This yields the following system of equations (see [18]):

$$\frac{\partial S_w}{\partial t} + \nabla \cdot \mathbf{v}_w = 0,$$

with 
$$\mathbf{v}_w = -\frac{k_w}{\mu_w} \nabla p_w$$

$$\frac{\partial S_n}{\partial t} + \nabla \cdot \mathbf{v}_n = 0,$$

with 
$$\mathbf{v}_n = -\frac{k_n}{\mu_n} \nabla p_n$$

$$\frac{\partial a_{wn}}{\partial t} + \nabla \cdot (a_{wn} \mathbf{w}_{wn}) = E_{wn},$$

with 
$$\mathbf{w}_{wn} = -\frac{k_{wn}}{a_{wn}} \nabla a_{wn}$$

$$S_w + S_n = 1$$

$$p_n - p_w = p_c$$

$$a_{wn} = a_{wn}(S_w, p_c),$$

where \(S_\alpha\) is the saturation, \(\mathbf{v}_\alpha\) the Darcy velocity, \(k_{\alpha}\) the relative permeability, \(\mu_\alpha\) the dynamic viscosity, and \(p_\alpha\) the pressure of the \(\alpha\) phase. Furthermore, \(t\)
denotes the time, $\phi$ is the porosity, $K$ stands for the tensor of intrinsic permeability. The specific interfacial area is denoted by $a_{wn}$, $v_{wn}$ is the interfacial velocity, $K_{wn}$ is the interfacial permeability, while $E_{wn}$ is a general production rate term for the specific interfacial area. Finally, $p_c$ denotes the capillary pressure.

The specific interfacial area and its production rate are depending on the primary variables $S_w$ and $p_c$. The relative permeabilities are only depending on $S_w$. The particular form of these expressions depends on the porous medium and is determined on experimental basis. The choices made in this work are discussed in Section 4; for further details we refer to [18].

The particular form of the production rate $E_{wn}$ plays an important role in constructing the semi-analytical solution to the interfacial area model. In the absence of any experimental information, we rely on physical considerations to define $E_{wn}$. Specifically, the rate of change of the interfacial area is high whenever significant changes in the saturation occur. Therefore we assume that the rate of change of the interfacial area is proportional to the rate of change of the wetting phase saturation. A first ansatz in this sense is

$$E_{wn} = -e_{wn} \frac{\partial S_w}{\partial t},$$

where $e_{wn}$ is a production function that depends on the primary variables, $S_w$ and $p_c$; a discussion on its specific form is postponed to Section 4.

The standard model consists of the phase balance equations (1) and (2), the closure condition (4), as well as the definition of the capillary pressure (5). Notice that in this case the capillary pressure depends only on the saturation. The extended model (1) - (6) involves a new unknown quantity, the interfacial area. In this paper we reduce the mathematical structure of the model (1) - (6) to the redistribution problem introduced by Philip [21], and then apply the procedure in [6] for constructing a solution to the resulting problem.

Recalling that the flow is one-dimensional, since the phase saturations add to a constant, the total velocity $v = v_w + v_n$ is constant in space. Here we assume the total velocity being constant in time as well. To model redistribution only, we set $v = 0$. Then the balance equations (1) - (2) are reduced to the scalar two phase model

$$\phi \frac{\partial S_w}{\partial t} + K \frac{\partial}{\partial x} \left\{ \frac{k_{rw}}{\mu_w} \mu_n \left( \frac{k_{rw}}{\mu_w} + \frac{k_{rn}}{\mu_n} \right)^{-1} \frac{\partial p_c}{\partial x} \right\} = 0. \tag{8}$$

**Remark 2.1** The same approach can be considered for the case of unsaturated flow, when the non-wetting phase is gaseous (e.g. air) and assumed at a constant pressure, say $p_{air} = 0$. Then we have $p_c = -p_w$ and $v_n = 0$. In this case the balance equations reduce to

$$\phi \frac{\partial S_w}{\partial t} + K \frac{\partial}{\partial x} \left\{ \frac{k_{rw}}{\mu_w} \frac{\partial p_c}{\partial x} \right\} = 0. \tag{9}$$
2.2 The dimensionless form

We put equations (3) - (8) in a dimensionless form. Let $L$, $T$, $P$, and $A$ be characteristic values for the length, time, capillary pressure, and interfacial area, respectively, and transform the variables and unknowns by

$$
x := \frac{x}{L}, \quad t := \frac{t}{T}, \quad p := \frac{p}{P}, \quad a = \frac{a_{wn}}{A}, \quad e = \frac{e_{wn}}{A}.
$$

(10)

We choose $T$ and $L$ such that

$$
\frac{T}{L^2} = \Phi \frac{\mu_w}{KP}.
$$

(11)

Furthermore, we introduce the viscosity ratio $M = \frac{\mu_w}{\mu_n}$, as well as the interfacial number

$$
C_{ia} = \Phi \frac{\mu_n AK_{wn}}{KP}.
$$

(12)

Disregarding the subscript in the water saturation $S_w$, we end up with the dimensionless model

$$
\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left( \frac{k_{rw}(S) k_{rn}(S)}{MK_{rw}(S) + k_{rn}(S)} \frac{\partial p}{\partial x} \right) = 0,
$$

(13)

$$
\frac{\partial a}{\partial t} - C_{ia} \frac{\partial}{\partial x} \left( \frac{a}{a_{wn}} \right) = -e \frac{\partial S}{\partial t},
$$

(14)

for all $x \in \mathbb{R}$ and $t > 0$. Initial conditions have to be specified for completing the model:

$$
S(x, 0) = S_0(x) \quad \text{and} \quad p(x, 0) = p_0(x) \quad \text{for all} \quad x \in \mathbb{R}.
$$

(15)

Note that the initial conditions are given for the capillary pressure, and not for the interfacial area. Knowing $S_0$ and $p_0$, we can use (6) to obtain the initial specific interfacial area $a_0$.

In view of Remark 2.1, the dimensionless model for unsaturated flow replaces equation (13) by

$$
\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left( k_{rw}(S) \frac{\partial p}{\partial x} \right) = 0.
$$

(16)

Defining the nonlinear function $D(\cdot)$ as

$$
D(S) = \begin{cases} 
\frac{k_{rw}(S) k_{rn}(S)}{k_{rw}(S) + Mk_{rn}(S)}, & \text{for the two-phase model,} \\
\frac{k_{rw}(S)}{k_{rw}(S) + Mk_{rn}(S)}, & \text{for the unsaturated flow model,}
\end{cases}
$$

(17)

we can combine both balance equations (13) and (16) into the single form

$$
\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left( D(S) \frac{\partial p}{\partial x} \right) = 0.
$$

(18)
Based on experimental investigations in flow cells (see [5]), one can conclude that the interfacial number is small. It typically ranges between $10^{-6}$ and $10^{-4}$. This allows us to disregard the diffusion term in the interfacial balance equation. In other words, the parabolic equation (14) is replaced by the ordinary differential equation

$$\frac{\partial a}{\partial t} = -e \frac{\partial S}{\partial t}. \tag{19}$$

The system is closed by (6), i.e. $a = a(S,p)$.

3 The semi-analytical solution

In this section we discuss the construction of a semi-analytic approach for the model (18) and (19). We start by constructing appropriate capillary pressure-saturation curves, and then seek for self-similar solutions in a simplified context.

3.1 The capillary pressure-saturation curves

Inserting (6) into (19) gives

$$a_S \frac{\partial S}{\partial t} + a_p \frac{\partial p}{\partial t} = -e \frac{\partial S}{\partial t}, \tag{20}$$

where the subscripts in $a_\alpha$ denote partial differentiation with respect to the argument $\alpha$ ($\alpha = S, p$).

In Section 4 we give an explicit form for the function $a(S,p)$, which is based on preliminary investigations on pore network models (see [15]). In the relevant range for $S$ and $p$ we have $a_p < 0$, as well as $e + a_S < 0$. \tag{21}

Then for any arbitrarily fixed $x \in \mathbb{R}$ we get:

$$\frac{\partial p}{\partial t} = -\frac{e + a_S}{a_p} \frac{\partial S}{\partial t}. \tag{22}$$

Since $x$ is fixed, we can interpret $p$ as a function of $S$ and obtain

$$\frac{dp}{dS} = -\frac{e + a_S}{a_p}. \tag{22}$$

The initial condition associated to (22) is provided by the initial conditions (15). Specifically, for the fixed $x$ and at $t = 0$ we have $p = p_0(x)$ and $S = S_0(x)$, implying

$$p(S_0) = p_0. \tag{23}$$

In this way we end up with a family of initial value problems depending on the parameter $x \in \mathbb{R}$:

$$(IVP)_x \quad \begin{cases} \frac{dp}{dS} = -\frac{e + a_S}{a_p}, \\ p(S_0(x)) = p_0(x). \end{cases} \tag{24}$$
Solving these problems for any \( x \in \mathbb{R} \) would provide the capillary pressure - saturation curve at that point, \( p(x, t) = p(S(x, t); x) \). Notice that both \( t \) and \( x \) are acting only as parameters.

In view of (21), the functions on the right in (24) are smooth in both arguments \( S \) and \( p \). Standard theory for ordinary differential equations ensures the existence and uniqueness of a solution for Problem \((IVP_x)\). This property has two immediate consequences. First, any pair \((S, p)\) defines a unique capillary pressure - saturation curve satisfying \( p(S) = p \). In particular, if the pair \((S, p)\) lies on a bounding capillary pressure - saturation curve (drainage or imbibition), then drainage or imbibition will follow along that curve. Furthermore, the uniqueness for Problem \((IVP_x)\) also implies that two different \( p - S \) curves can never intersect. In particular, a secondary \( p - S \) curve cannot cross any of the two bounding curves, and therefore it will stay inside the domain defined by these. In Section 4 below we take this into account when constructing the production function \( e \).

Next, the existence and uniqueness of a solution also shows a limitation of the ansatz in (7) for the interfacial area production term. Specifically, there is a unique capillary pressure - saturation curve through a given point, regardless of the process taking place: drainage or imbibition. In other words, the assumed form for \( E \) and \( e \) does not make any distinction between drainage and imbibition. A possible remedy is to let \( e \) depend also on the sign of \( \partial_t S \), allowing then for two different \( p - S \) curves corresponding to either drainage or imbibition processes. However, here we seek for a solution to the redistribution problem, where drainage is encountered in one subdomain, whereas imbibition appears in the other one. Therefore the shortcoming of (7) has no effect in the present context.

**Remark 3.1** The definition of Problem \((IVP_x)\) is based on the interfacial area equation (19). Consequently, along any capillary pressure - saturation curve solving \((IVP_x)\), (19) will be satisfied automatically. We will use this remark in the next section, where the interfacial area equation equation (19) is disregarded when constructing a solution to our model.

### 3.2 The similarity solution

In this section we construct a self-similar solution of the interfacial area model by considering special initial data for saturation and capillary pressure. Specifically, we assume that initially both \( S \) and \( p \) are constant to the left and to the right of \( x = 0 \):

\[
S_0(x) = \begin{cases} 
S_L, & \text{if } x < 0, \\
S_R, & \text{if } x > 0
\end{cases} \quad \text{and} \quad p_0(x) = \begin{cases} 
p_L, & \text{if } x < 0, \\
p_R, & \text{if } x > 0
\end{cases}
\tag{25}
\]

Notice that in each subdomain, both the initial saturation and initial pressure are constant. However, a pressure gradient is encountered at the interface \( x = 0 \), which causes flow from one subdomain to the other one. As a consequence, drainage takes place in one of the subdomains and imbibition in the
other one. Without loss of generality we assume that the initial conditions are chosen such that drainage occurs for \( x < 0 \) and imbibition for \( x > 0 \).

As follows from (24), only two capillary pressure - saturation curves have to be computed: \( p^{-} = p^{-}(S) \), obtained for the initial data \( p^{-}(S_{l}) = p_{l} \) for all \( x < 0 \), and \( p^{+} = p^{+}(S) \) with \( p^{+}(S_{r}) = p_{r} \) for all \( x > 0 \). In view of Remark 3.1, along any of these curves, the interfacial area equation (19) is satisfied automatically. Thus having determined the two curves \( p^{\pm}(S) \), we can disregard (19) and reduce the original model to

\[
\begin{cases}
\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left( D(S) \frac{\partial p^{-}(S)}{\partial x} \right) = 0, \\
S(x, 0) = S_{l},
\end{cases}
\]

for \( x < 0 \), respectively

\[
\begin{cases}
\frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \left( D(S) \frac{\partial p^{+}(S)}{\partial x} \right) = 0, \\
S(x, 0) = S_{r},
\end{cases}
\]

for \( x > 0 \).

### 3.2.1 The interface conditions

The two submodels are coupled at the interface \( x = 0 \), where we impose continuity of water flux pressure. Introducing the right and the left limits of the saturation at \( x = 0 \):

\[
S^{-}(t) = S(0-, t), \quad \text{and} \quad S^{+}(t) = S(0+, t),
\]

the coupling conditions become

\[
D(S^{-}(t)) \frac{\partial p^{-}(S^{-}(t))}{\partial x} = D(S^{+}(t)) \frac{\partial p^{+}(S^{+}(t))}{\partial x},
\]

and

\[
p^{-}(S^{-}(t)) = p^{+}(S^{+}(t)).
\]

**Remark 3.2** In this framework we have reduced the interfacial area model (18) - (19) to the redistribution problem posed by Philip ([21]). The major difference here is due to the interfacial area. Whereas in the Philip problem two pressure - saturation curves are specified a-priori (one for drainage, respectively one for imbibition), in the present approach we use the interfacial area and the initial data to determine these curves.

We proceed by seeking for self-similar solutions of (26) and (27). As in [2, 3, 6, 7, 21] we set

\[
S = S(\eta) \quad \text{and} \quad p = p(\eta) \quad \text{with} \quad \eta = \frac{x}{\sqrt{t}}.
\]
and obtain the problems
\[
(SP_\ell) \quad \begin{cases} \frac{\eta^2}{2} S' + (D(S) p^- (S'))' = 0, & \text{for } \eta \in (-\infty, 0), \\ S(-\infty) = S_\ell, \end{cases}
\]
and
\[
(SP_r) \quad \begin{cases} \frac{\eta^2}{2} S' + (D(S) p^+ (S'))' = 0, & \text{for } \eta \in (0, +\infty), \\ S(+\infty) = S_r. \end{cases}
\]

The two subproblems are coupled at the interface \( \eta = 0 \) by the conditions (29) and (30). To express these conditions in terms of the similarity variable \( \eta \) we use the unknown limits
\[
S^- = \lim_{\eta \to 0^-} S(\eta), \quad \text{and} \quad S^+ = \lim_{\eta \to 0^+} S(\eta),
\]
and define the left and right fluxes at \( \eta = 0 \):
\[
F^-(S^-) = D(S^-) \frac{dp^-}{dS} \bigg|_{S=S^-} \lim_{\eta \to 0^-} S'(\eta),
\]
\[
F^+(S^+) = D(S^+) \frac{dp^+}{dS} \bigg|_{S=S^+} \lim_{\eta \to 0^+} S'(\eta).
\]
In this way the conditions at the interface become
\[
F^-(S^-) = F^+(S^+), \quad \text{respectively} \quad p^-(S^-) = p^+(S^+).
\]

3.2.2 The left and right limits \( S^\pm \)

The main step in solving the subproblems \((SP_{\ell, r})\) is to determine the left and the right limits \( S^\pm \) of the saturation at \( \eta = 0 \). To do so we follow the construction proposed in [6] for the redistribution problem in the case of a porous column involving two different permeabilities. The procedure is based on results for similarity solutions in semi-infinite, as well as infinite domains, obtained in [2, 3, 7]. We start mentioning the following

**Theorem 3.3 (Existence and uniqueness):**

Given \( S^- < S_\ell \), Problem \((SP_\ell)\) with \( S(0^-) = S^- \) has a unique solution, which is decreasing. Similarly, any \( S^+ > S_r \) uniquely determines a (decreasing) solution to Problem \((SP_r)\) satisfying \( S(0^+) = S^+ \).

As proven in the papers mentioned above, an \( \eta_l \in [-\infty, 0) \) exists defining a maximal interval \((\eta_l, 0)\) on which \( S \) is strictly decreasing. Furthermore, if \( \eta_l > -\infty \), then \( S(\eta) = S_\ell \) for all \( \eta \leq \eta_l \). Analogous, \( S \) is strictly decreasing on \((0, \eta_r)\), whereas \( S(\eta) = S_r \) for all \( \eta \geq \eta_r \) whenever \( \eta_r < \infty \). Necessary and sufficient conditions for the finiteness of \( \eta_l \) and \( \eta_r \) are given in [2, 3, 7].

Based on Theorem 3.3, we discuss the initial conditions (25). As assumed in the beginning of Section 3.2, drainage occurs for \( x(\eta) < 0 \) and imbibition for
The case strictly increasing functions $h^\pm$ are decreasing, the pressure continuity at $\eta = 0$ gives:

$$p^-(S_l) \leq p^-(S^-) = p^+(S^+) \leq p^+(S_r).$$

The case $p^-(S_l) = p^+(S_r)$ is trivial since the solution is constant in each subdomain. Therefore we restrict ourselves to the cases when the inequalities in the above are strict, yielding a necessary condition for the solvability of the coupled $(SP_{l,r})$ problems:

$$p^-(S_l) < p^+(S_r).$$  \hspace{1cm} (36)

Furthermore, $p^\pm$ are strictly monotone, thus invertible. We introduce the strictly increasing functions $h_+^\pm, h_-^\pm : [0,1] \rightarrow [0,1]$

$$h_+^\pm(s) := (p^+)^{-1}(p^-(s)), \text{ respectively } h_-^\pm(s) := (p^-)^{-1}(p^+(s)), \hspace{1cm} (37)$$

for any $s \in [0,1]$. Then the pressure continuity at $\eta = 0$, as well as (36), become

$$S^+ = h_+^+(S_-), \quad \text{ or } \quad S^- = h_-^-(S_+), \quad \text{ respectively}$$

$$S_l > h_-^+(S_r), \quad \text{ or } \quad S_r < h_+^-(S_l).$$  \hspace{1cm} (38)

Using these notations we can reformulate the results in [7]:

**Theorem 3.4 (Monotonicity of $F^\pm$):**

The flux $F^-$ is increasing in $S^-$ for any $h_+^-(S_r) \leq S^- \leq S_\ell$. Furthermore,

$$F^-(S_\ell) = 0, \quad \text{and} \quad F^-(h_+^-(S_r)) > 0.$$  \hspace{1cm} (39)

Similarly, $F^+$ is decreasing in $S^+$ for any $h_+^+(S_\ell) \geq S^+ \geq S_r$, with

$$F^+(S_r) = 0, \quad \text{and} \quad F^+(h_+^+(S_\ell)) > 0.$$  \hspace{1cm} (40)

Both fluxes $F^\pm$ depend continuously on the arguments.

Theorem 3.4 immediately implies the existence and uniqueness of a solution pair $(S^-, S^+)$ satisfying the interface conditions (35). To see this we use first the pressure condition for writing $S^+ = h_+^+(S^-)$ and seek for an $S^- \in [h_+^-(S_r), S_\ell]$ yielding the flux continuity at the interface. With

$$\mathcal{F} : [h_+^-(S_r), S_\ell] \rightarrow \mathbb{R}, \quad \mathcal{F}(s) = F^-(s) - F^+(h_+^+(s)),$$  \hspace{1cm} (39)

we have defined a continuous and increasing function satisfying

$$\mathcal{F}(S_\ell) = F^-(S_\ell) - F^+(h_+^+(S_\ell)) < 0, \quad \text{and}$$

$$\mathcal{F}(h_+^+(S_r)) = F^-(h_+^+(S_r)) - F^+(S_r) > 0,$$  \hspace{1cm} (40)

implying the existence of a unique $s_0 \in (h_+^-(S_r), S_\ell)$ solving $\mathcal{F}(s_0) = 0$. Taking $S^- = s_0$ and $S^+ = h_+^+(s_0)$ we have obtained the left and right limits of the self-similar saturation at the interface $\eta = 0$. 

12
The above existence result suggests a straightforward numerical approach for calculating the saturation pair \((S^-, S^+)\), the bisection. Such an approach uses, however, the fluxes \(F^\pm(\cdot)\) evaluated for different arguments. In a direct approach, given a saturation pair \((s, \bar{s})\) satisfying the pressure condition \(s = h^+(\bar{s})\) (see (38)), one can use (34) to compute \(F^-(s)\) and \(F^+(\bar{s})\). This requires solving (31) on the semi-infinite intervals \((0, \infty)\) and with \(S(-\infty) = S_\ell\), \(S(0-) = s\), as well as the similar equation (32) on \((0, \infty)\), where \(S(0+) = \bar{s}\) and \(S(\infty) = S_r\). Notice that in this way we determine the saturation \(S\) on the entire real axis and for any pair of values \((s, \bar{s})\), whereas only \(F^-(s)\) and \(F^+(\bar{s})\) are needed. The solution \(S\) can be computed later and only for the correct values \(S^\pm\).

To reduce the computational effort we consider the approach in [6] that is based on the monotonicity of \(S\) in \(\eta\). For any given pair \((s, \bar{s})\) we can invert \(S\): \((-\eta_\ell, 0) \rightarrow (s, S_\ell)\) and define \(\eta: (s, S_\ell) \rightarrow (-\eta_\ell, 0)\). This gives \(S'(\eta(S)) = 1/\eta'(S)\), yielding \(F^-(S) = D(S) \frac{d\eta}{dS} \frac{1}{\eta'(S)}\). Therefore (31) can be transformed into

\[
\frac{d}{dS} (F^-(S)) = \frac{\eta}{2} \quad \text{for } s < S < S_\ell.
\] (41)

In the limit \(S \searrow s\) we get \(\frac{dF^-}{dS}(S = s) = 0\), whereas \(F^-(S_\ell) = 0\) by Theorem 3.4. Using these conditions and differentiating (41) in \(S\) gives

\[
(FP_\ell) \begin{cases} F^-(S) \frac{d^2}{dS^2} (F^-(S)) = \frac{1}{2} D(S) \frac{dp^-}{dS}, & \text{for } s < S < S_\ell, \\
\frac{dF^-(s)}{dS} = 0, & F^-(S_\ell) = 0. \end{cases}
\] (42)

In a similar way, on the positive subinterval we have

\[
(FP_r) \begin{cases} F^+(S) \frac{d^2}{dS^2} (F^+(S)) = \frac{1}{2} D(S) \frac{dp^+}{dS}, & \text{for } S_r < S < \bar{s}, \\
\frac{dF^+(\bar{s})}{dS} = 0, & F^+(S_\ell) = 0. \end{cases}
\] (43)

The problems above can be solved for any pair of values \((s, \bar{s})\) satisfying the pressure condition, yielding the left and right fluxes \(F^-(s)\) and \(F^+(\bar{s})\). In view of (40), there exists a unique pair yielding the flux continuity at \(\eta = 0\). This pair also represents the sought saturation limits \(S^\pm\). To determine these two values we consider the following algorithm, which is based on the conditions at interface (35), as well as Theorem 3.4.

Determine the curves \(p^\pm(\cdot)\) by solving the problem \((IVP_\pm)\) with the initial data (25).

Let \(S_\ell := h^+(S_\ell)\) and \(S_r := S_\ell\).

While \(S_\ell > \bar{s} > TOL\) do (TOL being a given tolerance)

\[s := (S_\ell + \bar{s})/2.\]
Solve Problem \((FP_\ell)\), giving \(F^-(s)\).
Solve Problem \((FP_r)\) with \(\overline{s} = h^+(s)\), giving \(F^+(\overline{s})\).
Compute \(\mathcal{F}(s)\) as given in (39). If \(\mathcal{F}(s) > 0\) let \(S_\ell := s\). If \(\mathcal{F}(s) < 0\) let \(S_\ell := s\).
Go to the beginning of the "while" loop.

Take \(S^- = s\) and \(S^+ = h^+(s)\).

In the above algorithm we have \(S_\ell \geq S_\ell\). This is ensured by the theorems 3.3 and 3.4, as well as by the monotonicity of \(h^+\) and \(h^-\).

Having determined \(S^\pm\) we can find a solution to the problems \((SP_\ell,r)\). To do so we follow [6] and use the fluxes \(F^\pm\) solving the problems \((FP_\ell,r)\). Specifically, (41) immediately gives

\[
\eta(S) = \begin{cases} 
    2 \frac{d}{ds}(F^+(S)) > 0, & \text{if } S_\ell < S < S^+ \\
    2 \frac{d}{ds}(F^-(S)) < 0, & \text{if } S^- < S < S_\ell. 
\end{cases}
\]  

Clearly, the solution \(S : \mathbb{R} \setminus \{0\} \rightarrow (S_\ell, S^+) \cup (S^-, S_\ell)\) is determined by inverting \(\eta\) given above. At \(\eta = 0\) this solution is completed by the left and right limits \(S(0-) = S^-\) and \(S(0+) = S^+\).

4 An example

In this section we present an example for the redistribution problem including interfacial area effects. For the numerical calculations we have considered quadratic relative permeabilities. In the dimensionless framework they become \(k_{rw}(S) = S^2\) and \(k_{rn}(S) = (1 - S)^2\). For the bounding capillary pressure - saturation curves we have chosen power functions with negative and sub-unitary arguments: \(p_{dr}^c(S) = S^{-\lambda_D}\) with \(\lambda_D = 2\) for the bounding drainage curve, and \(p_{imb}^c(S) = S^{-\lambda_I}\) with \(\lambda_I = 4\) for the bounding imbibition curve (see Figure 3).

4.1 The functions \(a\) and \(e\)

For the interfacial area equation (6) we consider a bi-quadratic dependency in \(S\) and \(p\),

\[
a = a(S,p) = a_{00} + a_{10}S + a_{01}p + a_{20}S^2 + a_{02}p^2 + a_{11}Sp. \tag{45}
\]

This form has been used in numerical studies based on pore-scale network models (see [15]), where appropriately chosen coefficients have led to a good agreement with the numerical data. In the dimensionless setting, the parameters determined there become

\[
a_{00} = -0.0358, a_{10} = 0.5535, a_{01} = 0.0510, a_{20} = -0.3937, a_{11} = -0.1842, \text{ and } a_{02} = -0.1872.
\]
In this case we have \( a_p(S, p) < 0 \) for any pair \((S, p)\) inside the region of the \( S - p \) plane that is determined by the bounding capillary pressure - saturation curves.

Unfortunately neither experimental nor numerical data is available yet for the production function \( e \) in (19). In choosing a particular form for \( e \) one should account for two important situations. These occur when the saturation - pressure pair are moving along the bounding drainage function \( p_{dr}(S) \), as well as along the bounding imbibition function \( p_{imb}(S) \). Since these curves are bounding, the model should rule out a situation when the \((S, p)\) pair leaves the domain in the \( S - p \) plane that is comprised between these curves. Proceeding as in (20), we obtain

\[
e(S, p) = -a_p(S, p) \frac{dp}{dS}(S) - a_S(S, p)
\]  

along the known curves \( p_{dr}^c(\cdot) \) and \( p_{imb}^c(\cdot) \). To extend the definition of \( e \) to the entire region of interest we use linear interpolation along constant saturation. Specifically, given a pressure \( p \) we can determine uniquely two saturation values \( S_{dr} \) and \( S_{imb} \) satisfying \( p_{dr}^c(S_{dr}) = p \), respectively \( p_{imb}^c(S_{imb}) = p \). This defines

\[
e_{dr}(p) = -a_p(S_{dr}, p) \frac{dp_{dr}^c}{dS}(S_{dr}) - a_S(S_{dr}, p), \quad \text{respectively}
\]

\[
e_{imb}(p) = -a_p(S_{imb}, p) \frac{dp_{imb}^c}{dS}(S_{imb}) - a_S(S_{imb}, p).
\]

Interpolating between the two curves we define

\[
\bar{e}(S, p) = e_{imb}(p) - e_{dr}(p)
\]

and

\[
\bar{e}(S, p) = e_{imb}(p) - e_{dr}(p).
\]

At this point we remark that along any curve, the capillary pressure is a decreasing function of the saturation. Recalling (22) and since \( a_p < 0 \) the sum \( e + a_S \) should remain negative for any point \((S, p)\) between the two bounding curves. Observe that this condition is satisfied automatically along these curves. Therefore we define

\[
e(S, p) = \min\{0, \bar{e}(S, p)\}.
\]

4.2 A redistribution example

Initially we assume the following values for the left and right subdomains

\[
S_l = 0.8 \text{ and } p_l = 1.10, \quad \text{respectively} \quad S_r = 0.2 \text{ and } p_r = 1.68
\]

We have used this data to determine the corresponding drainage, respectively imbibition curves. These are (numerical) solutions to the problem \((IVP)_x\), where we have used the expressions in (46) and (48) for the interfacial area, respectively the production rate. The outer curves in Figure 3 are the bounding capillary pressure - saturation curves. The interior left curve is the capillary pressure saturation curve passing through \((S_r, p_r)\) in the imbibition subdomain
Figure 3: The capillary pressures: the outer left and right curves are the bounding imbibition and drainage curves. The interior curves are the ones corresponding to the initial data.

\( \eta > 0 \). Similarly, the interior right curve corresponds to the initial pair \((S_l, p_l)\) in the drainage subdomain \( \eta < 0 \).

The fluxes in Figure 4 are numerical solutions to the problems \((FP_r)\), respectively \((FP_l)\). The right and left saturation limits at \( \eta = 0 \) are providing continuity for the capillary pressure \( (p^- (S^-) = p^+ (S^+)) \), as well as for the flux across the interface. Applying the scheme described in the previous section we have obtained \( S^+ = 0.3654 \) and \( S^- = 0.4984 \). Correspondingly, at the interface we obtain \( F^- = F^+ = 0.0648 \), respectively \( p^- = p^+ = 1.3840 \).

Figure 5 displays the saturation in the similarity coordinate \( \eta = \frac{x}{\sqrt{t}} \). Drainage occurs in the left subdomain, where the saturation decays from the initial value \( S_l = 0.8 \) to \( S^- = 0.4984 \). On the right imbibition is encountered, with \( S \) decaying from \( S^+ = 0.3654 \) to \( S_r = 0.2 \).

Finally the pressure is displayed in Figure 6 in the similarity coordinate \( \eta \). Notice the continuity of the pressure at the interface. Furthermore, since the saturation is decreasing in both subdomains, the capillary pressure is increasing from \( p_l = 1.10 \) to \( p_r = 1.68 \).

5 Summary and outlook

We have considered a thermodynamically motivated model of two-phase flow in porous media that was recently treated in [18], and derived a semi-analytical solution for a wetting phase problem. Specifically, we have considered a homogeneous one-dimensional domain divided into two semi-infinite subdomains. The two phases have a constant initial saturation in each of the subdomains, with a high wetting-phase saturation in one part and a low wetting-phase saturation in the other part. Consequently, drainage takes place in the initially high saturation subdomain, whereas imbibition takes place in the initially low saturation.
Figure 4: The imbibition (left) and drainage (right) fluxes as functions of the saturation. Here $S^+ = 0.3654$ and $S^- = 0.4984$.

Figure 5: The selfsimilar saturation in the imbibition (left) and drainage (right) subdomains. Here $S_t = 0.8$, $S^- = 0.4984$, $S^+ = 0.3654$ and $S_r = 0.2$. 
part. These subdomains are separated by a fixed drainage–imbibition interface. The coupling conditions at the interface (flux and pressure continuity) lead to a persistent jump in the saturation. Based on the ideas in [21] and [6], we construct similarity solutions to the redistribution problem for the interfacial area model.

The interfacial area model is also a step towards hysteresis, where the interfacial area–saturation–capillary pressure surface can replace the drainage and imbibition scanning curves, which are numerous and difficult to include in a consistent mathematical model. Further research will be done in both experimental and mathematical direction. Experimental work is essential for having realistic models, where the parameters have to be determined and validated. Semi-analytical solutions are desirable also in other situations, where either one or both subdomains are finite. Furthermore, appropriate numerical algorithms have to be developed to perform realistic simulations for the interfacial area based model.

Acknowledgments

The authors are members of the International Research Training Group NUPUS, financed by the German Research Foundation (GRK 1398) and The Netherlands Organisation for Scientific Research (DN 81-754). The work of C.J. van Duijn and I.S. Pop was supported by the Dutch government through the national program BSIK: knowledge and research capacity, in the ICT project BRICKS (http://www.bsiik-bricks.nl), theme MSV1. J. Niessner’s work was funded by the German Research Foundation (DFG).
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