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SUMMARY

In this paper, we show the convergence of a multipoint flux approximation (MPFA) L-method. To do so, we introduce a boundary modification of the original MPFA L-method. Due to this local modification, we obtain the equivalence between our MPFA L-method and a conforming finite element approach with modified right hand side. The influence of the modified right hand side can be easily analyzed within the abstract framework of variational crimes.

KEY WORDS: multipoint flux; L-method; conforming finite element; a priori error estimate; homogeneous media

1. INTRODUCTION

Finite volume methods are numerical discretization techniques which can locally inherit physical conservation laws of the original problem. The property of discrete local mass conservation is desirable for many application fields, especially in computational fluid mechanics. Therefore, it is popular for multiphase flow in reservoir simulation.

The classical cell-centered finite volume (CCFV) method is a physically intuitive control-volume formulation using two-point flux approximation (TPFA), which is generally used to approximate elliptic operators in reservoir simulation. However, it is only correct if the grid directions are aligned with the principal directions of the permeability tensor $K$. For general non-$K$-orthogonal grids, TPFA does not work properly due to the error in its solution which cannot be reduced by refining the grids, see [1]. In reservoir simulation, grids with a high aspect ratio are quite often used, and grids with a more complex geometry are preferred at faults or in near-well regions. Thus, the multipoint flux approximation (MPFA) methods were...
widely studied in the last decade. It can give a correct discretization of flow equations not only for general non-orthogonal grids but also for general orientation of the principal directions of the permeability tensor.

There are many variants of the MPFA method, the most popular of which is the O-method. An introduction to the MPFA O-method for quadrilaterals can be found in [2–7]. The convergence of the O-method is discussed in [5, 8–10] by showing numerical results and in [11–13] from the theoretical point of view.

Another important property for multiphase flow is the monotonicity of the numerical elliptic operator which can avoid unphysical oscillations in the discrete solution. Conditions for monotonicity of the MPFA method on quadrilaterals are discussed in [14]. Based on the optimal monotonicity criteria derived in [14], a new MPFA method called the L-method was first introduced for quadrilateral grids in 2D in [15], and it was extended to 3D in [16]. The L-method has three main advantages compared to the O-method: smaller flux stencils, a larger domain of convergence and a larger domain of monotonicity. In [17], several aspects of the MPFA L-method on quadrilaterals are studied including the boundary influence on the convergence rate, and the geometrical interpretation of the original transmissibility-based criterion in the case of homogeneous media which yield more insight into the L-method. To the authors’ knowledge, there has not been a study for the error analysis of the MPFA L-method from the mathematical point of view. Thus, in this work, we focus on the theoretical study of the L-method and its a priori error estimates.

The rest of the paper is organized as follows. In Section 2, a modified MPFA L-method is proposed for the second-order elliptic problem with homogeneous media on uniform grids. We start from a brief illustration of the original L-method for the interior domain and then give the modifications of the L-method for the boundary. Some auxiliary theorems for the error analysis are also stated and proven in this section. The equivalence between the modified L-method and a modified conforming finite element method is built up in Section 3, which is the key point of the work and provides the basis for the theoretical study of the L-method. In Section 4, we present and prove the optimal order $H^1$ and $L^2$ error estimates. Finally, some conclusions are given in Section 5.

2. THE MODIFIED MPFA L-METHOD

In this section, a modified MPFA L-method is presented. For interior grid cells (see Figure 1), the MPFA L-method is the same as the original one introduced in [15] which is briefly illustrated in Subsection 2.1 for a model problem with homogeneous media on uniform grids. In this case, the L-method has seven-point cell stencils as mentioned in [15]. Here, we state it in a theorem and prove it using the observations and results for the L-method in [17]. In Subsection 2.2, we propose the modifications of the L-method for both Neumann and Dirichlet boundary grid cells (see Figure 1) which give a new boundary treatment of the L-method. Figure 1 shows the definition of different kinds of cells, where short and long corner cells denote the corner cells along the short and long diagonal direction, respectively.
Throughout this paper, we consider the second-order elliptic problem
\[
\begin{align*}
-\nabla \cdot (K \nabla p) &= q, \quad \text{in } \Omega \in \mathbb{R}^2, \\
K \nabla p \cdot n &= g_N, \quad \text{on } \Gamma_N, \\
p &= 0, \quad \text{on } \Gamma_D,
\end{align*}
\]
for homogeneous media on uniform grids in 2D. Here, \( \Omega \) is a bounded convex domain in \( \mathbb{R}^2 \) with a polygonal boundary \( \partial \Omega \). \( \Gamma_N, \Gamma_D \) are the Neumann and Dirichlet boundary part, respectively.

For simplicity of notation, we only consider the elliptic problem with pure Neumann or pure Dirichlet boundary conditions, i.e., \( \Gamma_N = \partial \Omega \) or \( \Gamma_D = \partial \Omega \). In the case of a pure Neumann boundary value problem, we additionally assume that surface and volume forces are compatible, i.e., \( \int_{\Gamma_N} g_N ds + \int_\Omega q dx = 0 \) and that \( \int_\Omega p dx = 0 \), which can guarantee the existence and uniqueness of its solution. \( K \) is the conductivity tensor and is assumed to be symmetric, positive definite and constant on \( \Omega \). In the context of reservoir simulation, problem (1) is the pressure equation for single phase flow in porous media. Correspondingly, \( K \) is the ratio of the intrinsic permeability tensor and the fluid viscosity, and \( p \) represents the pressure. As shown in Figure 2, the uniform grids can be uniform rectangular grids or uniform parallelograms.

The discretization lengths in x- and y-direction can be different, that is, \( h_x \neq h_y \) is allowed. It should be mentioned that uniform rectangular grids are suitable for a rectangular domain and uniform parallelogram grids can only be used for a parallelogram region.

2.1. Original MPFA L-method for interior grid cells
The MPFA L-method for problem (1) on a quadrilateral grid was introduced and well described in [15]. For completeness, here we shortly illustrate the L-method for the interior cells (see Figure 1) of a parallelogram grid in the following three steps:
(i) Choose a proper L triangle, as shown in Figure 3, either L triangle 1 or L triangle 2 to calculate the transmissibility coefficients $t_j^i$ of a half edge, such as the top half edge $\overline{x_1x_3}$ between cell 1 and cell 2 in Figure 3. Here, L triangle 1 and L triangle 2 are shortly denoted as $T_1$ and $T_2$, respectively; $j = 1, 2$ indicates that $T_j$ is applied in the computation; $i$ is the cell index, as shown in the figure, $i = 1, \cdots, 4$; $x_i, i = 1, \cdots, 4$, are the cell centers; $\bar{x}_k, k = 1, 2, 3$, are midpoints of the edges; and $\bar{x}_3$ is the common corner of the four cells. Let the pressure value at the cell center $x_i$ be $p_i$, and the pressure values at $\bar{x}_k, k = 1, 2, 3$ are denoted by $\bar{p}_k$. In each of the three subcells of a L triangle, for example in $T_1$, which are the quadrilateral $x_1\bar{x}_1\bar{x}_3\bar{x}_2$, $\triangle \bar{x}_1x_2\bar{x}_3$ and $\triangle \bar{x}_2x_3\bar{x}_1$, linear pressure functions are applied. And the pressure values $\bar{p}_k, k = 1, 2, 3$ can be eliminated in the expression of the flux through each half edge by two continuity conditions: (a) full pressure continuity, (b) flux continuity at the two interfaces (such as, in $T_1$, $\overline{x_1x_3}$ and $\overline{x_2x_3}$) inside a L triangle. Finally, the flux through each half edge can be explicitly expressed by the linear combination of the pressures at the three cell centers which form the L triangle with the transmissibility coefficients; that is, for $T_1$, $f = t_1^1p_1 + t_3^2p_2 + t_2^3p_3$, and for $T_2$, $f = t_1^2p_1 + t_3^2p_2 + t_2^3p_4$. For the detailed deduction of the flux expression, see [15]. Actually, the flux calculation for each half edge can be significantly simplified for homogeneous media, see Theorem 3.1 in [17]. The criterion for choosing the proper L triangle is: if $|t_1^1| < |t_2^2|$, choose $T_1$; otherwise, choose $T_2$. An intuitive and geometrical interpretation of the criterion can be found in [17], which will be later used for proofs.

![Figure 3](image3.png)

**Figure 3.** L triangle 1 ($T_1$, left) and L triangle 2 ($T_2$, right) for the MPFA L-method.

(ii) Use the procedure in (i) to calculate the fluxes $f_1^i$ and $f_2^i$ through two half edges, then the flux $f_1$ through an entire edge is obtained by the sum $f_1^1 + f_1^2$. Similar to the other three edges of a cell $K \in T_h^*$, $T_h^*$ is the partition of the domain $\Omega$, i.e., the MPFA mesh.

![Figure 4](image4.png)

(iii) Insert the four flux expressions derived in (ii) into the local control-volume formulations $f_1 + f_2 + f_3 + f_4 = \int_K q dx$ for all cells $K \in T_h^*$; the discretization of the MPFA L-method is derived.

In Figure 4, the quadrilateral $x_1\overline{x_2x_3x_4}$ is a parallelogram cell in a uniform grid. $n_1$ and $t_1$ are the outer normal and a unit tangential vector of the cell edge $\overline{x_3x_4}$, and $n_2$ and $t_2$ are the outer normal and a unit tangential vector of the cell edge $\overline{x_2x_3}$. The orientation of the
tangential vector $t_1$ and $t_2$ is fixed such that both point toward $x_3$. The shadowed region is denoted as Region 2, and the remaining part of the plane is Region 1. Then, the following lemma holds for any symmetric and positive definite tensor $K$.

![Figure 4. A parallelogram cell with the normal and tangential vectors of two cell edges.](image)

**Lemma 2.1**

Let $K$ be a symmetric and positive definite tensor. Then the vectors $K^T n_1$ and $K^T n_2$ are both in the same region, either Region 1 or Region 2 in Figure 4.

**Proof**

Obviously, there hold $n_1 = -R t_1 = R^T t_1$ and $n_2 = R t_2$, where $R$ is a rotation matrix with an angle $-\frac{\pi}{2}$, i.e.

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hfill (2)

Hence, we have

$$n_1^T t_2 = t_1^T R t_2 = t_1^T n_2.$$  \hfill (3)

Observe that $t_1$ and $t_2$ are linear independent, we find suitable coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$K^T n_1 = \alpha t_1 + \beta t_2,$$  \hfill (4)

$$K^T n_2 = \gamma t_1 + \delta t_2.$$  \hfill (5)

Thus, to prove $K^T n_1$ and $K^T n_2$ are both in Region 1, it is equivalent to prove that

(I) if $\alpha, \beta$ have the same sign, then $\gamma, \delta$ have the same sign;

Similarly, to prove $K^T n_1$ and $K^T n_2$ are both in Region 2, it is equivalent to prove that

(II) if $\alpha, \beta$ have opposite signs, then $\gamma, \delta$ have opposite signs.

Multiplying (4) and (5) by $n_1$ and by $n_2$, respectively, we find due to the fact that $K$ is symmetric and positive definite

$$0 < n_1^T K^T n_1 = \beta n_1^T t_2,$$  \hfill (6)
Now it is easy to see that $\beta, \gamma$ have the same sign. Since $n_1^T K^T n_2 = n_2^T K^T n_1$, we get $\alpha = \delta$ and thus $\alpha$ and $\delta$ have the same sign. Therefore, if $\alpha, \beta$ have the same sign, then $\gamma, \delta$ have the same sign; similarly, (II) also holds.

\[ n_1^T K^T n_2 = \delta n_2^T t_2, \quad (7) \]
\[ n_2^T K^T n_1 = \alpha n_1^T t_2, \quad (8) \]
\[ 0 < n_2^T K^T n_2 = \gamma t_1^T n_2 = \gamma n_1^T t_2. \quad (9) \]

Theorem 2.1

For homogeneous media and uniform parallelogram grids, the MPFA L-method has a seven-point cell stencil for the discretization of each interior cell $K \in \mathcal{T}_h^*$, i.e., the discretization of each cell $K$ is a seven-point scheme with the pressure values at the centers of seven cells including the center cell $K$ and six closest neighboring cells, as shown in Figure 5 (right).

Proof

Denote the permeability tensor of the homogeneous media by $K$, and let $n_1$ and $n_2$ be the outer normal of the cell edge $y_3y_4$ and $y_2y_3$, respectively. Assume that the direction of $K^T n_1$ is given, it is enough to prove that only a seven-point cell stencil is applied for the discretization of the interior cell $K$ in Figure 5.

According to the Illustration 2 in Subsection 3.2.2 of [17] and the given direction of $K^T n_1$, it is easy to see that the L triangle $\triangle x_i x_1 x_2$ is chosen for the half cell edge $y_3x_3$ as shown in Figure 5 (left). Hence, the L triangle $\triangle x_i x_2 x_3$ is correspondingly applied for the other half cell edge $y_4x_3$ using Theorem 3.1 in [17]. Since the cell edge $y_1y_2$ is parallel with $y_3y_4$, similarly, the L triangles $\triangle x_i x_4 x_5$ and $\triangle x_i x_5 x_6$ are used for the half edges $y_1x_1$ and $y_2x_1$, respectively (see Figure 5 (middle)).

Finally, from Lemma 2.1 we have that the vector $K^T n_2$ locates in the same region as $K^T n_1$. Therefore, in terms of the observations and results in [17], the L triangles $\triangle x_i x_1 x_2$ and $\triangle x_i x_1 x_6$ are chosen for the cell edge $y_2y_5$ and $\triangle x_i x_3 x_4$ and $\triangle x_i x_4 x_5$ are chosen for the cell edge $y_1y_4$, respectively, as shown in Figure 5 (right). \hfill \Box

2.2. A modified MPFA L-method for boundary grid cells

In this subsection, the modifications of the L-method for the flux calculations of Neumann and Dirichlet boundary cells are described, respectively. The modification in the case of Dirichlet boundary is quite attractive and may overcome a weak point of the original L-method. In the original L-method, the Dirichlet boundary is not easy to handle. One has to combine
the original L-approach with the O-method (see [17]) or to introduce fictitious ghost cells on the exterior. The modification for the Neumann boundary is not so natural and is mainly motivated by the fact to obtain an equivalent finite element approach.

In our modifications, whether for the Neumann or Dirichlet boundary cell, the L triangles for the half edges near and on the boundary are constructed in a different way. For example, for the boundary cell \( y_1y_3y_4y_6 \) in Figure 6 (left), two L triangles \( \triangle x_1x_3 \) and \( \triangle x_2x_3 \) are chosen for the flux calculation of the edges \( y_2y_3 \) and \( y_3y_4 \) instead of using one L triangle constructed with the ghost cells on the exterior in the original L-method. Similarly, \( \triangle x_3x_4 \) and \( \triangle x_4x_5 \) are applied for the edges \( y_3y_4 \) and \( y_4y_5 \). Thereby, seven L triangles with dotted lines shown in Figure 6 (left) are derived. Here, we divide the boundary cell \( y_1y_3y_4y_6 \) into two local cells \( y_3y_4y_8 \) and \( y_1y_6y_8 \), denoted as \( K_1 \) and \( K_2 \), respectively, see Figure 6 (right).

In contrast to the original method, we compute three fluxes for the edge \( y_4y_6 \) and \( y_1y_3 \). More precisely, the half edges \( y_1y_5 \) and \( y_3y_7 \) are decomposed into two subedges, and for each of these subedges one flux is computed.

2.2.1. Neumann boundary In terms of the above new construction of the L triangles, we can see that for an edge cell near the Neumann boundary, such as the parallelogram \( y_1y_3y_4y_6 \) in Figure 6 (left), not only the information at the cell centers \( x_1, x_5, x_6, x_7, x_i \) but also the information at the Neumann boundary points \( x_2, x_3, x_4 \) are needed for the flux calculation of the modified L-method. Correspondingly, our modification is to calculate two cell flux stencils for \( K_1 \) and \( K_2 \) separately instead of one cell flux stencil for \( y_1y_3y_4y_6 \). Therefore, we get two solution equations for the Neumann boundary cell, one corresponding to the cell center \( x_i \) and one related to the Neumann boundary point \( x_3 \).

In detail, for the local cell \( K_2 \), we calculate its cell flux stencil using the six L triangles.
\( \Delta x_1 x_7, \Delta x_6 x_7, \Delta x_5 x_6, \Delta x_4 x_5, \Delta x_3 x_4 \) and \( \Delta x_1 x_3 \). For the local cell \( K_1 \), the L triangle \( \triangle x_1 x_2 x_3 \) and \( \Delta x_1 x_3 \) are used to calculate the fluxes \( f_{38}, \hat{f} \) through the edges \( y_3 y_8 \) and \( y_8 y_{10} \), respectively; and the L triangle \( \triangle x_3 x_4, x_4 \) is applied for the fluxes \( f_{49}, \bar{f} \) through the edges \( y_4 y_9 \) and \( y_9 y_{10} \), see Figure 7. Finally, the fluxes \( f_{33}, f_{34} \) through the two half edges \( y_3 y_4 \) and \( y_4 y_5 \) are given by the Neumann boundary condition, i.e., \( f_{33} = - \int_{y_3 y_4} g_N ds, f_{34} = - \int_{y_4 y_5} g_N ds \).

![Figure 8. The L triangle \( \triangle x_1 x_2 x_3 \) near the Neumann boundary.](image)

Since the pressure \( p \) can be expressed as a linear function in each L triangle of the modified L-method such as \( \triangle x_1 x_2 x_3 \) for homogeneous media (see Lemma 3.1 in [17]), the gradient expression of the pressure in \( \triangle x_1 x_2 x_3 \) may be written as

\[
\nabla p = - \frac{1}{2F} [(p_2 - p_1) \nu_2 + (p_3 - p_1) \nu_3],
\]

(10)

which is deduced in [15]. Here, \( F \) is the area of \( \triangle x_1 x_2 x_3 \); \( p_i \) is the pressure value at the point \( x_i, i = 1, 2, 3 \); \( \nu_i \) is the scaled outer normal vector on the edge \( e \) lying opposite to the point \( x_i \), having the same length as \( e \), see Figure 8. Hence, the flux through the half edge \( x_1 x_2 \) in the L-method is approximated by

\[
f_{12} = - K \nabla p \cdot \nu_3,
\]

(11)

and it holds for all the fluxes through the interior edges of the interior and boundary cells.

**Remark**

Note that for this modification of the Neumann boundary treatment, the pressure values at the Neumann boundary points cannot be completely eliminated by continuity conditions and the Neumann boundary condition for each Neumann boundary cell as in the original L-method. In other words, the unknowns of the modified L-method contains the pressure values at all cell centers and all the Neumann boundary points which are in fact the Neumann boundary nodes of the finite element mesh introduced in the next section. Although the freedoms of the modified L-method increases compared to the original method, this extra effort can be neglected for large problems.

2.2.2. Dirichlet boundary  To deal with the flux calculation of an edge cell near the Dirichlet boundary, such as the parallelogram \( y_1 y_3 y_4 y_6 \) in Figure 6 (left), we only calculate one cell flux
stencil for it using the seven L triangles shown in Figure 6 (left) instead of two stencils for the Neumann boundary cell.

The difference with the original L-method is to handle the flux $f_{23}$ through $y_{2}y_{5}$ by the sum of the fluxes $f_{28}, f_{38}$ through $y_{2}y_{8}$ and $y_{3}y_{8}$ using the two L triangles $\triangle x_{1}x_{3}$ and $\triangle x_{1}x_{2}x_{3}$, respectively, similar to the flux through $y_{4}y_{5}$.

![Figure 9. Local cell $K_{1}$ near the Dirichlet boundary and its fluxes.](image)

For the cell edge on the Dirichlet boundary, such as $y_{3}y_{4}$ in Figure 6 (left), a special treatment is given to calculate the flux $\tilde{f}_{34}$ through it, see Figure 9. First, the fluxes $\hat{f}, \bar{f}, f_{38}, f_{49}$ are calculated as described in the modification for the Neumann boundary using the expressions (10) and (11). Then, the flux through $y_{3}y_{4}$ is defined by

$$\tilde{f}_{34} = \int_{K_{1}} q \, dx - (\hat{f} + \bar{f}) - (f_{38} + f_{49}).$$

(12)

**Remark**

For the corner cells with either the Neumann or Dirichlet boundary condition, the modification of the L-method for their calculations of the flux stencils is in principle similar as that for the edge cells introduced in either Subsubsection 2.2.1 or 2.2.2, but a bit more complicated.

![Figure 10. The modified L-method for a corner cell at the Neumann (left) and Dirichlet (right) boundary.](image)

In brief, the modification for the corner cells is also given in order to build up the equivalence between the modified MPFA L-method and a finite element method. For the short corner cell at the Neumann boundary, the modification is to calculate four subcell flux stencils, see Figure 10 (left), and its fluxes through the Neumann boundary are still given by the Neumann boundary condition. For the short corner cell at the Dirichlet boundary, one cell flux stencil is calculated with the new artificial definition of the fluxes $\tilde{f}_{37}, \tilde{f}_{38}$ through two Dirichlet boundary segments $x_{3}x_{7}$ and $x_{3}x_{8}$, as shown in Figure 10 (right). For the long corner cells at either the Neumann or Dirichlet boundary, the modifications are the same as for the short corner cells.
3. THE EQUIVALENCE BETWEEN THE MODIFIED L-METHOD AND A MODIFIED CONFORMING FINITE ELEMENT METHOD

In this section, the equivalence between the modified MPFA L-method introduced in Section 2 and a modified conforming finite element method (FEM) is presented and proven in the case of homogeneous media and uniform grids. In terms of this equivalence, a priori estimates for the MPFA L-method can be established.

From Theorem 2.1 and the boundary modifications of the L-method, it can be seen that all the L triangles for the modified MPFA L-discretization form a quasi-uniform triangular finite element mesh as shown in Figure 11. The triangulation of the finite element mesh is denoted as $T_h$.

Let $N^*_h$ denote the index set of all interior grid nodes of $T_h$, which are actually all cell centers of the MPFA mesh $T^*_h$. In detail, $N^*_h$ consists of two disjoint index sets, i.e., $N^*_h = N^{i}_h \cup N^{b}_h$, where $N^{i}_h$ and $N^{b}_h$ are the index sets of all cell centers of the interior cells $i$ and boundary cells $b$ in $T^*_h$ (see Figure 1), respectively. Denote $N^{i}_h = N^{i,e}_h \cup N^{i,c}_h \cup N^{i,l}_h$, where $N^{i,e}_h$, $N^{i,c}_h$ and $N^{i,l}_h$ as shown in Figure 12 are the index sets of all cell centers of the edge cells $e$, short and long corner cells $c_e$, $c_l$, respectively. Here, $N^{i}_h = N^{c,e}_h \cup N^{c,l}_h$ denotes the index sets of all cell centers of the corner cells. The index set of all Neumann and Dirichlet boundary nodes of $T_h$ are denoted by $N^{N}_h$ and $N^{D}_h$, respectively, and let $N_h = N^*_h \cup N^{N}_h \cup N^{D}_h$. We remark that for the pure Neumann problem, $N^{D}_h = \emptyset$, while for the pure Dirichlet problem, $N^{N}_h = \emptyset$. Let $N^N_h = N^{N,c}_h \cup N^{N,s}_h \cup N^{N,l}_h \cup N^{N,c_e}_h \cup N^{N,c_l}_h$, where $N^{N,c}_h$, $N^{N,s}_h$ and $N^{N,l}_h$ are the index sets of all Neumann boundary nodes locating in the middle of the boundary segment of the edge cells, short and long corner cells, respectively; $N^{N,c_e}_h$ and $N^{N,c_l}_h$ are the index sets of all Neumann boundary nodes at the short and long corners of the domain, respectively, see Figure 12.

![Figure 11. MPFA mesh with solid lines and triangular finite element mesh with dotted lines.](image1)

![Figure 12. Nodes’ definition of the finite element mesh.](image2)
Let $V_h$ be the standard linear finite element space defined on the triangulation $T_h$

$$V_h = \{ v \in C(\Omega) : v|_T \text{ is linear for all } T \in T_h \text{ and } v|_{\Gamma_D} = 0 \text{ if } \Gamma_D \neq \emptyset \}. $$

Let $\phi_i, i \in N_h \setminus N_h^{D}$, be the standard nodal basis functions associated with the node $x_i$, which are piecewise linear functions on $\Omega$ defined by

$$\phi_i(x_l) = \delta_{il} = \begin{cases} 1, & \text{if } l = i, \\ 0, & \text{otherwise,} \end{cases}, \quad i \in N_h \setminus N_h^{D}, \quad l \in N_h. \quad (13)$$

As shown in Figures 13-15, the local support $\text{supp } \phi_i$ of the basis function $\phi_i, i \in N_h^e$ is the polygon $x_1x_2x_3x_4x_5x_6$; the local support $\text{supp } \phi_i$ of $\phi_i, i \in N_h^{N,e} \cup N_h^{N,s} \cup N_h^{N,l}$ is the polygon $x_1x_2x_3x_4$; the local support $\text{supp } \phi_i$ of $\phi_i, i \in N_h^{N,c_e}$ is the polygon $x_1x_2x_3x_4$; and the local support $\text{supp } \phi_i$ of $\phi_i, i \in N_h^{N,c_l}$ is the polygon $x_1x_2x_3$. Obviously, $V_h$ can be constructed by the linear combination of the basis functions, i.e., $V_h := \text{span}\{\phi_i, i \in N_h \setminus N_h^{D}\}$. We define the following interpolation operators $I_h : C(\Omega) \rightarrow V_h$:

$$I_h v = \sum_{i \in N_h \setminus N_h^{D}} v(x_i) \phi_i(x), \quad (14)$$
and $I_h^*:\quad I_h^*v = \sum_{i \in \mathcal{N}_h \setminus \mathcal{N}_h^0} v(x_i) I_h^* \phi_i(x), \quad (15)$

where $I_h^* \phi_i(x) = \chi_{D_i}(x) = \begin{cases} 1, & \text{if } x \in D_i, \\ 0, & \text{otherwise,} \end{cases}$ \quad (16)

where $D_i = K_i$ if $i \in \mathcal{N}_h^i$; $D_i = K_2$, if $i \in \mathcal{N}_h^c$; $D_i = K_1$, if $i \in \mathcal{N}_h^{N,e}$, as illustrated in Figure 13.

$I_h^* \phi_i(x) = \begin{cases} 1, & \text{if } x \in D_i^1, \\ \phi_i, & \text{if } x \in D_i^2, \\ 0, & \text{otherwise,} \end{cases}$ \quad (17)

where $D_i^1 = L_i$ if $i \in \mathcal{N}_h^{N,e} \cup \mathcal{N}_h^{c_i}$; $D_i^1 = K_5$, if $i \in \mathcal{N}_h^{c_i} \cup \mathcal{N}_h^{N,l}$; $D_i^2 = K_3$, if $i \in \mathcal{N}_h^{c_i} \cup \mathcal{N}_h^{N,l}$; $D_i^3 = K_4$, if $i \in \mathcal{N}_h^{c_i} \cup \mathcal{N}_h^{N,l}$, as illustrated in Figures 14-15 (left) and (middle).

$I_h^* \phi_i(x) = \phi_i, \ x \in \text{supp} \phi_i, \ i \in \mathcal{N}_h^{C,e} \cup \mathcal{N}_h^{C,e_i}$, \quad (18)

as illustrated in Figures 14-15 (right). Let $I_h^*|_{\Gamma_N} = I_h^*|_{\Gamma_N}$, which is the trace of the interpolation operator $I_h^*$ on the Neumann boundary $\Gamma_N$. According to the definition of $I_h^*$, it is easy to see that the following relation

\[ \int_{\Omega} (\phi_i - I_h^* \phi_i) dx = 0 \quad (19) \]

holds for $i \in \mathcal{N}_h^* \cup \mathcal{N}_h^{N,e} \cup \mathcal{N}_h^{N,e_i} \cup \mathcal{N}_h^{N,e_i}$, but not for $i \in \mathcal{N}_h^{N,s} \cup \mathcal{N}_h^{N,l}$.

In order to prove the equivalence, let $p_h \in V_h$ be the solution of a modified conforming FEM defined by

\[ a(p_h, v_h) = (q, I_h^* v_h)_{\Omega} + (g_N, I_h^* v_h)_{\Gamma_N}, \quad \forall v_h \in V_h, \quad (20) \]

where $a(p_h, v_h) = \int_{\Omega} \mathbf{k} \nabla p_h \nabla v_h \, dx$, $(q, I_h^* v_h)_{\Omega} = \int_{\Omega} q I_h^* v_h \, dx$, $(g_N, I_h^* v_h)_{\Gamma_N} = \int_{\Gamma_N} g_N I_h^* v_h \, ds$. We note that $I_h^* 1 = 1$ and $I_h^* 1 = 1$, thus the compatibility condition on $g_N$ and $q$ guarantees the existence of a solution also in the discrete setting. Then the following theorem holds.

**Theorem 3.1**

For homogeneous media and uniform grids, the modified MPFA L-method is equivalent to (20) in the sense that $p_h(x_i) = p_i$, $i \in \mathcal{N}_h$, where $p_i$ are the pressure values of the modified MPFA L-method at all grid nodes of $I_h$; and the flux of the modified MPFA L-method through each half cell edge or edge segment $E$ except that on the boundary $f_E = -\mathbf{K} \nabla p_h \cdot \mathbf{n}_E |E|$, where $\mathbf{n}_E$ is the unit outer normal on $E$, $|E|$ is the length of $E$.

**Proof**

The proof can be completed by four steps:

C1: For an interior cell, take $v_h = \phi_i, i \in \mathcal{N}_h^i$ in (20), we have

\[ a(p_h, \phi_i) = (q, I_h^* \phi_i)_{\Omega}. \quad (21) \]
Referring to Figure 16, since $p_h$ is piecewise linear on $T_h$, $K$ is constant on $\Omega$ and $\phi_i = 0$ on the boundary of its support, we get

$$a(p_h, \phi_i) = \int_{\text{supp}\phi_i} K \nabla p_h \nabla \phi_i dx = \sum_{T \in \text{supp}\phi_i} \int_T K \nabla p_h \nabla \phi_i dx$$

$$= \sum_{T \in \text{supp}\phi_i} \left( \int_{\partial T} K \nabla p_h \cdot n \phi_i ds - \int_T \nabla \cdot (K \nabla p_h) \phi_i dx \right)$$

$$= \sum_{T \in \text{supp}\phi_i} \int_{\partial T} K \nabla p_h \cdot n \phi_i ds$$

$$= \sum_e \int_{\partial S} [K \nabla p_h \cdot n_e] \phi_i ds = \sum_e [K \nabla p_h \cdot n_e] e |e| \frac{|e|}{2}$$

$$= \sum_S \int_{\partial S} K \nabla p_h \cdot n ds - \sum_{E \in \partial K} \int_E K \nabla p_h \cdot n_E ds$$

$$= \sum_S \int_S \nabla \cdot (K \nabla p_h) dx - \sum_{E \in \partial K} \int_E K \nabla p_h \cdot n_E ds$$

$$= - \sum_{E \in \partial K} K \nabla p_h \cdot n_E |E|,$$

where $K \in T_h^*$ is the interior parallelogram cell, $T \in T_h \cap \text{supp}\phi_i$ are those triangles in the triangulation of the finite element mesh which construct the support of $\phi_i$, $S = T \cap K$, $e$ are the interior edges of the support $\text{supp}\phi_i$, $n_e$ is the unit normal on $e$, the orientation of which is arbitrary but fixed; $E = S \cap \partial K$ are the edge segments of each interior cell in the MPFA mesh, $n_E$ is the unit outer normal on $E$, and $|E|$ is the length of $E$; the jump $[v] := v|_{T_1} - v|_{T_2}$, see Figure 16 (right).

On the other hand, according to the definition of $I_h^*$, we have

$$(q, I_h^* \phi_i)_\Omega = \int_K q dx.$$  

(23)
Combining (21), (22) and (23), it is easy to see that (21) is equivalent to
\[ - \sum_{E \in \partial K} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E| = \int_K q dx, \] (24)
which is exactly the original MPFA L-method for the interior cell $K$.

C2: For an edge cell near the Neumann boundary, first take $v_h = \phi_i, i \in \mathcal{N}_h^e$ in (20), we have
\[ a(p_h, \phi_i) = (q, I_h^* \phi_i)_\Omega. \] (25)
Similar as the deduction of (22) and (23), combining with (25), we obtain
\[ - \sum_{E \in \partial K_2} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E| = \int_{K_2} q dx, \] (26)
where $K_2$ is shown in Figure 13 (middle).
Next take $v_h = \phi_i, i \in \mathcal{N}_h^{N,e}$ in (20), we have
\[ a(p_h, \phi_i) = (q, I_h^* \phi_i)_\Omega + (g_N, I_{N,G}^* \phi_i)_{\Gamma_N}. \] (27)
Similar as the deduction of (22), since $\phi_i \neq 0, i \in \mathcal{N}_h^{N,e}$ on the Neumann boundary, it holds that
\[ a(p_h, \phi_i) = \sum_{T \in \text{supp}\phi_i} \int_{\partial T} \mathbf{K} \nabla p_h \cdot \mathbf{n}_T \phi_i ds \]
\[ = \sum_{T} [\mathbf{K} \nabla p_h \cdot \mathbf{n}_T]|_e \frac{|e|}{2} \int_{\Gamma_N \cap \text{supp}\phi_i} \mathbf{K} \nabla p_h \cdot \mathbf{n}_T \phi_i ds \]
\[ = - \sum_{E \in \partial K_1 \setminus \Gamma_N} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E|, \] (28)
where $K_1$ is shown in Figure 13 (right). Combining (27) and (28), we get
\[ - \sum_{E \in \partial K_1 \setminus \Gamma_N} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E| - \int_{\partial K_1 \setminus \Gamma_N} g_N ds = \int_{K_1} q dx. \] (29)
From (26) and (29), it can be seen that (25) and (27) correspond to the modified MPFA L-method for the Neumann boundary cell $K$.

C3: For an edge cell near the Dirichlet boundary, take $v_h = \phi_i, i \in \mathcal{N}_h^e$ in (20), we have
\[ a(p_h, \phi_i) = (q, I_h^* \phi_i)_\Omega. \] (30)
Similar as the deduction of (22) and (23), combining with (30), we obtain
\[ - \sum_{E \in \partial K_2} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E| = \int_{K_2} q dx, \] (31)
where $K_2$ is shown in Figure 13 (middle). Then, we have
\[ - \sum_{E \in \partial K \setminus \Gamma_D} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E| + \sum_{E \in \partial K_1 \setminus \Gamma_D} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E|E| + \int_{K_1} q dx = \int_K q dx, \] (32)
where $K_1$ is shown in Figure 13 (right). Let $\tilde{f}$ be the flux through the cell edge on the Dirichlet boundary $\partial K \cap \Gamma_D$, which is defined in the same way as (12). Then from the definition of $\tilde{f}$, it is easy to see that

$$- \sum_{E \in \partial K \setminus \Gamma_D} \mathbf{K} \nabla p_h \cdot \mathbf{n}_E \mid E + \tilde{f} = \int_K q \, dx,$$

which corresponds to the modified MPFA L-method for the Dirichlet boundary cell $K$.

C4: For a corner cell near the Neumann or Dirichlet boundary, the equivalence can be proven as the case C2 or C3 in terms of the new modification of the L-method mentioned in the remark of Subsubsection 2.2.2 and the appropriate definition of the operator $\mathcal{I}_h^*$ (17), (18) for the corner cells.

Finally, since the finite element solution $p_h$ is piecewise linear on $\mathcal{T}_h$ which is in fact constructed by all the L triangles for the modified L-method, $\nabla p_h$ has the same expression as (10) for the L-method. Therefore, from (24), (26), (29) and (33), it is easy to see that $p_h(x_i) = p_i, i \in \mathcal{N}_h$, and the cell edge fluxes of the modified L-method follow the desired result. In this sense, the modified MPFA L-method is equivalent to the modified conforming FEM (20). \[ \square \]

Remark
The equivalence between the modified MPFA L-method and the modified FEM (20) can be easily extended to non-homogeneous Dirichlet boundary conditions and to problems with non-trivial $\Gamma_N$ and $\Gamma_D$.

### 4. ERROR ESTIMATE

In this section, we present and prove the $H^1$ and $L^2$ error estimates for the modified MPFA L-method of problem (1) with either pure Neumann or pure Dirichlet boundary by the equivalence with the modified FEM (20). The optimal error estimates are derived.

We shall use the standard notation as that in [18]. For any subdomain $D \subset \Omega$, we introduce $L^2(D)$, with inner product $(\cdot, \cdot)_D$ and norm $\| \cdot \|_{0,D}$, and introduce Sobolev space $H^s(D)$, with norm $\| \cdot \|_{s,D}$ and seminorm $| \cdot |_{s,D}$, where $s$ may be fractional. If $D = \Omega$, we simplify the notation as follows: $| \cdot |_{s,D} \equiv | \cdot |_s$, $\| \cdot \|_{s,D} \equiv \| \cdot \|_s(s > 0)$, $(\cdot, \cdot)_D \equiv (\cdot, \cdot), \| \cdot \|_{0,D} \equiv \| \cdot \|$. Moreover, $H^{\frac{s}{2}}(\Gamma_N)$ is the range of $H^s(\Omega)$ by the trace operator and $H^{-\frac{s}{2}}(\Gamma_N)$ is the dual space of $H^{\frac{s}{2}}(\Gamma_N)$ with $(\cdot, \cdot)_{\Gamma_N}$ being the duality pairing. In the following, we understand $H^{\frac{s}{2}}(\Gamma_N)$ as a broken $H^{\frac{s}{2}}$-space, i.e., $H^{\frac{s}{2}}(\Gamma_N) = \{ v \in L^2(\Gamma_N) : v \in H^{\frac{s}{2}}(\gamma_l), l = 1, \cdots, L \}$, where $\gamma_l$ are straight line segments and $\bigcup_{l=1}^L \gamma_l = \Gamma_N$, but $\gamma_l \cup \gamma_k, l \neq k$, is not a straight line. In this sense, we also define $\| \cdot \|_{\frac{s}{2}, \Gamma_N}$ as a broken norm, i.e., $\| v \|_{\frac{s}{2}, \Gamma_N} = \sum_{l=1}^L \| v \|_{\frac{s}{2}, \gamma_l}$, and $\| \cdot \|_{-\frac{s}{2}, \Gamma_N}$ stands for its dual norm.

#### 4.1. $H^1$ error estimate

**Theorem 4.1**

Assume that $p$ and $p_h$ are the solutions of problem (1) and the modified finite element scheme (20), respectively, $p \in H^2(\Omega)$ and $q \in L^2(\Omega)$. If $\Gamma_N \neq \emptyset$, we also assume $g_N \in H^{\frac{s}{2}}(\Gamma_N)$ and
that the compatibility condition is satisfied. Then there exists a positive constant \( C > 0 \) such that
\[
\|p - p_h\|_1 \leq Ch(\|p\|_2 + \|g\| + \|g_N\|_{\frac{1}{2}, \Gamma_N}).
\] (34)

**Proof**
According to the Strang lemma in [18], we have that for \( \forall v_h \in V_h \)
\[
\|p - p_h\|_1 \leq \inf_{v_h \in V_h} \|p - v_h\|_1 + \sup_{w_h \in V_h} \frac{(q, I_h^* w_h - w_h) + (g_N, I_h^* w_h - w_h)_{\Gamma_N}}{w_h}_1.
\] (35)

Here, according to the definition of the interpolation operator \( I_h \)
\[
\inf_{v_h \in V_h} \|p - v_h\|_1 \leq \|p - I_h p\|_1 \leq Ch\|p\|_2.
\] (36)

Using the definition of \( I_h^* \), we get
\[
(q, I_h^* w_h - w_h) \leq \|q\| \|I_h^* w_h - w_h\| \leq Ch\|q\| \|w_h\|_1.
\] (37)

Here, we have used that \( I_h^* \) is a locally defined operator preserving constants. Moreover, we have
\[
(g_N, I_h^* w_h - w_h)_{\Gamma_N} \leq \|g_N\|_{\frac{1}{2}, \Gamma_N} \|I_h^* w_h - w_h\|_{-\frac{1}{2}, \Gamma_N},
\] (38)

where from the definition of \( I_h^* \) and the trace theorem, it holds for \( u = I_h^* w_h - w_h \),
\[
\|u\|_{-\frac{1}{2}, \Gamma_N} = \sup_{0 \neq v \in H^\frac{1}{2}(\Gamma_N)} \frac{(u, v)_{\Gamma_N}}{\|v\|_{\frac{1}{2}, \Gamma_N}} = \sup_{0 \neq v \in H^\frac{1}{2}(\Gamma_N)} \frac{(u, v - v_h)_{\Gamma_N}}{\|v\|_{\frac{1}{2}, \Gamma_N}} \\
\leq Ch^{\frac{1}{2}} \|u\|_{0, \Gamma_N} = Ch^{\frac{1}{2}} \|I_h^* w_h - w_h\|_{0, \Gamma_N} \\
\leq Ch \|w_h\|_{\frac{1}{2}, \Gamma_N} \leq Ch \|w_h\|_1,
\] (39)

here, \( v_h \) is piecewise constant on the partition of the Neumann boundary, and the fact that \( (I_h^* w_h - w_h, v_h)_{\Gamma_N} = 0 \) is used.

Thus, from (38) and (39), we obtain
\[
(g_N, I_h^* w_h - w_h)_{\Gamma_N} \leq Ch \|g_N\|_{\frac{1}{2}, \Gamma_N} \|w_h\|_1.
\] (40)

Combining (35), (36), (37) and (40), the desired result follows. \( \square \)

4.2. \( L^2 \) error estimate

In this subsection, an auxiliary basis is constructed for the \( L^2 \) error analysis. The optimal \( L^2 \) error estimates are derived for both pure Dirichlet and pure Neumann boundary problems.

Now we construct the new basis \( \psi_i, i \in \mathcal{N}_h^* \) such that
1) \( \sum_{i \in \mathcal{N}_h^*} \psi_i = 1; \)
2) \( \text{supp} \psi_i = \text{supp} \phi_i, i \in \mathcal{N}_h^* \setminus \mathcal{N}_h^c; \)
3) \( \|\psi_i\| \leq Ch; \)
4) \( \int_\Omega \psi_i dx \geq Ch^2; \)
5) \( \int_\Omega \psi_i(\phi_j - I_h^* \phi_j) dx = 0, i, j \in \mathcal{N}_h^*. \)
Using 1), 2) and 5) and setting $\psi_i$ to be piecewise constant, a straightforward calculation yields $\psi_i$ for the different types of finite element nodes as illustrated in Figures 17-18.

For the above basis $\psi_i$, we can define the following interpolation operator $P_h : L^2(\Omega) \rightarrow W_h := \text{span}\{\psi_i, i \in N_h^*\}$

$$P_h v = \sum_{i \in N_h^*} \alpha_i \psi_i, \quad \alpha_i = \frac{\int_{\Omega} v \psi_i dx}{\int_{\Omega} \psi_i dx}.$$

(41)

Then, we have the following two properties of $P_h$: I) $P_h$ is $L^2$-stable; II) $|v - P_h v| \leq Ch|v|_1$.

**Theorem 4.2**

Assume that $p$ and $p_h$ are the solutions of problem (1) with the pure Dirichlet boundary and the modified finite element scheme (20), respectively, $p \in H^2(\Omega)$ and $q \in H^1(\Omega)$. Then there exists a positive constant $C > 0$ such that

$$\|p - p_h\| \leq C h^2 (\|p\|_2 + \|q\|_1) \leq Ch^2|q|_1.$$  

(42)

**Proof**

We consider the adjoint problem of (1): find $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\begin{cases}
- \nabla \cdot (K \nabla \varphi) = p - p_h, & \text{in } \Omega \subset \mathbb{R}^2, \\
\varphi = 0, & \text{on } \partial \Omega.
\end{cases}$$

(43)

It is well known that (43) has a unique solution, and moreover, due to the convexity of $\Omega$, the following regularity estimate:

$$\|\varphi\|_2 \leq C \|p - p_h\|$$  

(44)

holds. Using the Aubin-Nitsche technique and the definitions of the original problem (1), the discrete problem (20) and the adjoint problem (43), we get for $\forall \varphi_h \in V_h$

$$\|p - p_h\|^2 = a(p - p_h, \varphi - \varphi_h) + (q, \varphi_h - I_h^* \varphi_h).$$

(45)
Here, from Theorem 4.1 and the property of $I_h$, it is easy to get with $\phi_h = I_h \phi$

\[
a(p - p_h, \varphi - \varphi_h) \leq Ch(\|p\|_2 + \|q\|)\|\varphi - I_h \varphi\|_1
\leq Ch^2(\|p\|_2 + \|q\|)\|\varphi\|_2
\leq Ch^2\|q\|\|\varphi\|_2.
\] (46)

From the construction of the basis $\psi_i$ and the property of $P_h$, it holds that for $\forall \psi_h \in W_h$

\[
(q, \varphi_h - I_h^* \varphi_h) = (q - \psi_h, I_h \varphi - I_h^* I_h \varphi)
\leq \inf_{\psi_h \in W_h} \|q - \psi_h\| \|I_h \varphi - I_h^* I_h \varphi\|
\leq \|q - P_h q\| \|I_h \varphi - I_h^* I_h \varphi\|
\leq Ch^2|q|_1 \|I_h \varphi\|_1
\leq Ch^2|q|_1 \|\varphi\|_2.
\] (47)

Combining (44)-(47), it is easy to see that (42) holds.

Remark

For the pure Neumann boundary case, the basis $\psi_i$ for all the interior nodes $x_i, i \in N_h^i$ are exactly the same as for the Dirichlet case. But for the construction of the basis $\psi_i, i \in N_h \setminus N_h^i$, it gets more complicated due to the basis functions of the Neumann boundary nodes. It has been checked that the basis $\psi_i, i \in N_h^\text{N} \cup N_h^{\text{N}\text{c}}$ can be given in terms of the requirements of $\psi_i$. However, for all the corners, it is quite difficult to find the suitable $\psi_i$ and therefore, we handle the error analysis of the Neumann boundary case without considering the corners by simply assuming the right hand side $q \in H^1(\Omega) \cap L^\infty(\omega_c)$, where $\omega_c$ is the union of all corner cells and its neighbors.

The main difference for the proof of the $L^2$ error estimate for the Neumann boundary case is to handle the term $(q, \varphi_h - I_h^* \varphi_h)$ using the new basis $\psi_i$ without corners, i.e.,

\[
(q, \varphi_h - I_h^* \varphi_h) = (q - \psi_h, I_h \varphi - I_h^* I_h \varphi)_{\Omega \setminus \omega_c} + (q, I_h \varphi - I_h^* I_h \varphi)_{\omega_c}
\leq Ch^2|q|_1 \|\varphi\|_2 + Ch(|q|_{\infty, \omega_c} \|I_h \varphi - I_h^* I_h \varphi\|_0, \omega_c)
\leq Ch^2(|q|_1 + |q|_{\infty, \omega_c}) \|\varphi\|_2.
\] (48)

Based on (48), the optimal $L^2$ error estimate can also be derived for the pure Neumann boundary case, i.e., $\|p - p_h\| \leq Ch^2$ if the problem setting is regular enough.

5. CONCLUSIONS

This paper gives a theoretical study of the MPFA L-method by finding out the interesting relation between the modified L-method and a modified conforming finite element method. Although it only works for the special case with homogeneous media and uniform grids, it is quite helpful to further understand the L-method and is a direct way to prove the error estimates through the abstract framework for the finite element method.

We note that our techniques can be also applied to the more general case of an inhomogeneous permeability tensor and unstructured quadrilateral meshes as long as the chosen L triangles form a simplicial mesh, i.e., each L triangle is used for the computation of two half fluxes. However, then the proof is more technical and involves an estimate for the
consistency error of the associated nonconforming finite element approach. In that case, the finite element basis functions will be discontinuous at the midpoints of the cell edges and be defined on a submesh, see Figure 19 (left), and a reduced convergence rate might occur.

The submesh $T_h^s$ is obtained by cutting all L triangles with the mesh $T_h^s$ and introducing new inner edges by connecting the cell centers with suitable cell corners, see the dashed lines in Figure 19 (left).

Now the nonconforming finite element space $V_h^s$ is defined with respect to $T_h^s$

$$V_h^s = \{ v \in L^2(\Omega) : v|_{T_s} \in P_1(T_s), T_s \in T_h^s, \quad v \text{ is continuous at all nodes } x_i, i \in N_h, \quad v|_{T_m} \in C(T_m), \mathbf{K}\nabla v|_{T_m} \in H(\text{div}; T_m), T_m \in T_h^m \},$$

where $T_h^m$ is the macromesh, see Figure 19 (right). Each $T_m \in T_h^m$ is associated with one L triangle which is the union of four elements in $T_h^s$. Now $\dim V_h^s$ is equal to the number of cells and well defined. Moreover, in general, it defines a nonconforming finite element space.

It is important to give a sketch on what can be done for the more general case. We are pretty sure that from this we can get some more general results. The only need then is that each L triangle is used twice. And in the worst case, we can even modify the method again, and give for the stencil edge pair not all possibilities but only two.

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