“Non-linearities and upscaling in porous media“

Analysis of an upwind-mixed hybrid finite element method for transport problems

Fabian Brunner
Florin A. Radu
Peter Knabner

Preprint 2013/1
ANALYSIS OF AN UPWIND-MIXED HYBRID FINITE ELEMENT METHOD FOR TRANSPORT PROBLEMS

FABIAN BRUNNER*, FLORIN A. RADU†, AND PETER KNABNER‡

Abstract. We prove optimal order convergence of an upwind-mixed hybrid finite element scheme for linear parabolic advection-diffusion-reaction problems. It was introduced in [20] and is based on an Euler-implicit mixed hybrid finite element discretization of the problem in fully mass conservative form using the Raviart-Thomas mixed finite element of lowest order on triangular meshes. Optimal order convergence in time and space is obtained for the fully discrete formulation. The scheme provides the same order of convergence as the standard upwind-mixed method while it is more efficient since a local elimination of variables is possible with our choice of the upwind weights. The theoretical findings are sustained by a numerical experiment.

Key words. advection-diffusion problem, advection-dominance, mixed finite element method, upwind weighting, a priori error estimates

AMS subject classifications. 65M12, 65M15, 65M60, 76S05

1. Introduction. We analyze an upwind-mixed hybrid finite element approximation scheme for the linear parabolic advection-diffusion-reaction problem

\begin{align}
\partial_t c - \nabla \cdot (D \nabla c - Qc) + Rc &= f \quad \text{in } Q_T, \\
c &= c_0 \quad \text{on } \{0\} \times \Omega, \\
c &= 0 \quad \text{on } J \times \partial \Omega,
\end{align}

where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with sufficiently smooth boundary, \( T < \infty \) denotes the final time, \( J = (0, T) \) and \( Q_T = J \times \Omega \). This problem may be regarded as a model problem for a wide range of applications arising, e.g., in hydrology, civil engineering, petroleum engineering, economics and many other fields, and the need for accurate, efficient and reliable schemes to solve them numerically is well recognized. Despite of the simplicity of the problem and extensive research over the last decades, the design of such methods remains a challenging task, in particular when advection is strongly dominant. In this case, numerical instabilities typically occur when standard numerical methods are used since sharp layers in the solution cannot be resolved properly. The upwind-mixed scheme studied in this work was designed to improve the robustness of the standard mixed method for strongly advection-dominated problems while maintaining the efficiency of the classical method introduced in [12, 13], which is usually employed in its hybrid form to enable the use of iterative linear solvers [2, 4]. Generally, the interest of mixed methods lies in the fact that they are locally mass conservative and simultaneously provide accurate approximations of a scalar and a flux unknown. Moreover, the flux approximations are continuous across interelement boundaries.

There is a rich literature on mixed finite element methods and their convergence, see [4] and the references therein. Related to our paper are [2, 12, 13]. We also mention [1, 18, 19, 26] for convergence of mixed finite element methods for degenerate parabolic

*Department of Mathematics, University of Erlangen-Nuremberg, Cauerstrasse 11, 91058 Erlangen, Germany, brunner@math.fau.de
†Department of Mathematics, University of Bergen, Allégaten 41, 5007 Bergen, Norway, Florin.Radu@math.uib.no
‡Department of Mathematics, University of Erlangen-Nuremberg, Cauerstrasse 11, 91058 Erlangen, Germany, knabner@math.fau.de
problems and [17] for transport problems. A recent review on solving diffusion-type equations by mixed finite elements is presented in [27]. Upwind-mixed methods are treated in [8, 9] and in [20, 23]. Stabilization techniques based on using quadrature formulas for the mass matrix are employed in [16, 22].

The upwind-mixed scheme we study here is based on an Euler-implicit mixed hybrid finite element discretization using the Raviart-Thomas element of lowest order and was first introduced and studied numerically in [20] in the context of reactive transport simulation. Various numerical examples showed that the method works well and is robust for problems of practical interest involving high Péclet numbers. However, as all upwind methods, it introduces additional numerical diffusion leading to an artificial smearing of the numerical solution. We refer to the numerical experiments in [20], where the amount of artificial diffusion was quantified and compared to other discretization schemes. In this paper, we study the method analytically and establish optimal order convergence estimates. Further, we compare it to the upwind-mixed scheme of Dawson [8] with respect to accuracy and efficiency.

The definition of our upwind weights involves the Lagrange multipliers arising in the hybrid problem formulation and uses them to approximate the scalar unknown on adjacent cells. Consequently, these values are not needed explicitly to evaluate the upwind weighting formula and an efficient implementation of the method using a local elimination procedure remains possible. This is not the case when information from adjacent grid cells is used for the definition of the upwind weights. The idea of using Lagrange multipliers for the discretization of the advective term was also employed in [20, 23] for approximations with Raviart-Thomas elements of lowest order and in [5] for approximations with Brezzi-Douglas-Marini elements of lowest order. In the latter, optimal order convergence for the flux variable was observed numerically, whereas suboptimal order is obtained with the standard scheme when the flux variable is defined as the total flux consisting of diffusive and advective flux, cf. [10].

The article is structured as follows. In the next section, we state assumptions on the partial differential equation and the mesh and introduce the finite element spaces and projectors we are going to work with. In Sec. 3, the continuous variational problem associated with (1.1a)-(1.1c) is defined and sufficient conditions for existence and uniqueness are given. Moreover, the upwind-mixed hybrid scheme is presented and motivated. Sec. 4 contains the convergence analysis including existence and uniqueness for the fully discrete upwind-mixed hybrid scheme. In Sec. 5, numerical results are presented confirming the error estimates and comparing our method with the standard upwind-mixed method with respect to accuracy and efficiency. The paper ends with a conclusion.

2. Notations and assumptions. Throughout the following, the common notations of functional analysis are used. In particular, let \((\cdot, \cdot)\) denote the inner product on \(L^2(\Omega)\) or \((L^2(\Omega))^2\), respectively, and \(\|\cdot\|_0\) the corresponding norm. Further, \((\cdot, \cdot)\) indicates the inner product in \(\partial\Omega\) and \(\|\cdot\|_k\) stands for the norm in \(H^k(\Omega) = W^{k,2}(\Omega)\). If the inner products or norms are considered on a measurable subset \(K \subset \Omega\), an additional index \(K\) will be added, e.g. \(\|\cdot\|_{0,K}, (\cdot, \cdot)_K, (\cdot, \cdot)_{\partial K}\) etc. As usual, let \(H(\text{div}, \Omega)\) denote the space of functions in \((L^2(\Omega))^2\) having the divergence in \(L^2(\Omega)\). For the time discretization, \(N\) denotes a positive integer and we define the time step size \(\tau = T/N\) and the discrete times \(t^n = nt\). The time derivative is approximated by the backward difference quotient \(\frac{\partial c}{\partial t}^n = \frac{1}{\tau}(c^n - c^{n-1})\), where the superscript \(n\) indicates the evaluation of a function at the discrete time \(t = t^n\).
Assumptions on the partial differential equation. Throughout this work, we solve the equation (1.1a) along with the initial and boundary conditions (1.1b)-(1.1c). The following assumptions are made on the coefficient functions.

(A1) The coefficient matrix $D = (D_{ij})_{ij}$ is symmetric and $D_{ij} \in C(\overline{Q_T})$ for $i, j = 1, 2$. Furthermore, the uniform ellipticity condition

$$C_1 |\xi|^2 \leq \sum_{i,j=1}^{2} D_{ij}(t, x)\xi_i \xi_j \leq C_2 |\xi|^2$$

holds for all $\xi \in \mathbb{R}^2$, $(t, x) \in Q_T$ and given constants $C_1, C_2 > 0$.

(A2) Moreover, $Q \in C(J; (W^{1,\infty}(\Omega))^2)$, $R \in C(\overline{Q_T})$, $f \in C(J; L^2(\Omega))$ and $c_0 \in H^1_0(\Omega)$.

In addition to the regularity obtained by Theorem 3.1, we shall assume that the solution $(q, c)$ of the mixed variational problem (3.1)-(3.2) satisfies

(A4) $(q, c) \in C(J; (H^1(\Omega))^2) \times H^1(J; H^1(\Omega)) \cap H^2(J; L^2(\Omega))$.

Assumptions on the grid. Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of triangular decompositions of $\Omega$ such that

(M1) $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ and $K_1 \cap K_2 = \emptyset$ if $K_1 \neq K_2$, where the elements $K \in \mathcal{T}_h$ are closed triangles.

(M2) If $K_1 \cap K_2 = \emptyset$ and $K_1 \neq K_2$, then $K_1 \cap K_2$ is either a vertex or a full edge of each.

(M3) If $K \subset \Omega$, then $K$ has straight edges only.

(M4) If $K$ is a boundary triangle, the boundary edge can be curved.

(M5) $h_K = \text{diam}(K)$, max $h_K = h$.

(M6) $\{\mathcal{T}_h\}_{h>0}$ is shape-regular, i.e. there exists a constant $\sigma > 0$ such that $h_K \leq \sigma \rho_K$ for all $K \in \mathcal{T}_h$, where $\rho_K = \sup \{\text{diam}(S) : S$ is a disc in $\mathbb{R}^2$ and $S \subset K \}$ is the diameter of the inscribed circle of $K$.

The collection of all edges of an element $K$ is denoted by $\mathcal{E}(K)$, whereas $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D$ indicates the set of all edges of $\mathcal{T}_h$ consisting of the disjoint subsets $\mathcal{E}_h^I$ of interior edges and $\mathcal{E}_h^D$ of boundary edges.

Mixed approximation spaces and projections. Our upwind-mixed scheme is based on the Raviart-Thomas mixed finite element of lowest order. Hence, let

$$V_h = RT_0(\Omega, \mathcal{T}_h) = \{ v \in H(\text{div}, \Omega) : v|_K \in RT_0(K) \text{ for all } K \in \mathcal{T}_h \},$$

$$W_h = \{ w \in L^2(\Omega) : w|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathcal{T}_h \},$$

where $\mathcal{P}_0(K)$ denotes the restriction of polynomials of total degree not greater than $k$ to $K$ and $RT_0(K) = \{ v \in (L^2(K))^2 : v(x) = a + bx, \ a \in \mathbb{R}^2, \ b \in \mathbb{R} \}$

is the local Raviart-Thomas space of lowest order. Mixed finite element schemes may be rewritten equivalently in a hybrid formulation relaxing the continuity constraints of the flux variable across element interfaces and imposing them instead by introducing Lagrange multipliers and requiring additional variational equations. The approximation spaces associated with the mixed hybrid formulation are defined as

$$\tilde{V}_h = \{ v \in (L^2(\Omega))^2 : v|_K \in RT_0(K) \text{ for all } K \in \mathcal{T}_h \},$$

$$\Lambda_h = \{ \lambda \in L^2(\mathcal{E}_h) : \lambda|_E \in \mathcal{P}_0(E) \text{ for all } E \in \mathcal{E}_h^I, \lambda|_E = 0 \text{ for all } E \in \mathcal{E}_h^D \}.$$
In our convergence analysis, we make use of the usual projection operator \( \Pi \times P_h : (H^1(\Omega))^2 \times L^2(\Omega) \to V_h \times W_h [21] \), which can be extended to domains having a curved boundary, cf. [13]. The following properties hold for the projectors:

(i) \( P_h \) is the \( L^2(\Omega) \)-orthogonal projection onto \( W_h \);

(ii) for any \( v \in (H^1(\Omega))^2 \) and any \( w \in L^2(\Omega) \),

\[
(\nabla \cdot (v - \Pi_h v), w_h) = 0 \quad \text{for all } w_h \in W_h,
\]

\[
(\nabla \cdot v_h, w - P_h w) = 0 \quad \text{for all } v_h \in V_h;
\]

(iii) the following approximation properties hold:

\[
\| v - \Pi_h v \|_0 \leq C \| v \|_{1,h} \quad \text{for all } v \in (H^1(\Omega))^2,
\]

\[
\| w - P_h w \|_0 \leq C \| w \|_{1,h} \quad \text{for all } w \in H^1(\Omega).
\]

Here and in the following, \( C \) denotes a generic constant which is independent of the unknowns and the discretization parameters. The projection \( \Pi_h|_K \) is uniquely determined by means of moments over the edges,

\[
(\Pi_h|_K q - q) \cdot n_E, 1)_E = 0 \quad \text{for all } E \in \mathcal{E}(K),
\]

where \( n_E \) is the unit normal vector of the edge \( E \), outward to \( K \). In the same way, basis functions \( \{v_{KE}\}_{K \in T_h, E \in \mathcal{E}(K)} \) of \( \tilde{V}_h \) can be constructed with \( \text{supp}(v_{KE}) \subseteq K \) and

\[
(v_{KE} \cdot n_{E'}, 1)_{E'} = \delta_{EE'} \quad \text{for all } E, E' \in \mathcal{E}(K).
\]

For further use, let us mention that the basis functions are uniformly bounded in the \( L^2 \)-norm,

\[
\| v_{KE} \|_0 \leq C.
\]

If \( K \) is a straight-sided triangle, an explicit representation of the basis functions associated with \( K \in T_h \) is given by

\[
v_{KE}(x) = \frac{x - x_E}{2|K|} \chi_K(x), \quad E \in \mathcal{E}(K),
\]

where \( x_E \) denotes the corner point of \( K \) facing the edge \( E \), \( |K| \) is the measure of the element \( K \) and \( \chi_K \) the characteristic function of \( K \). Then, (2.3) is an immediate consequence of the shape-regularity (M6). For the spaces \( \Lambda_h \) and \( W_h \), basis functions are given by characteristic functions \( \{\mu_E\}_{E \in \mathcal{E}_h} \) of the interior edges and \( \{\chi_K\}_{K \in T_h} \) of the elements, respectively.

3. An Euler-implicit upwind-mixed hybrid finite element scheme of lowest order.

3.1. Continuous mixed variational formulation. Let us now formulate a continuous mixed variational problem associated with (1.1a)-(1.1c).

PROBLEM 1. Find \( (q, c) \in L^2(J; H(\text{div}, \Omega)) \times H^1(J; L^2(\Omega)) \) with \( c|_{t=0} = c_0 \) such that for almost every \( t \in J \)

\[
(D^{-1}q(t), v) - (\nabla \cdot v, c(t)) - (D^{-1}Qc(t), v) = 0,
\]

\[
(\partial_t c(t), w) + (\nabla \cdot q(t), w) + (Rc(t), w) = (f, w)
\]
for all \((v, w) \in H(\text{div}, \Omega) \times L^2(\Omega)\).

Note that here and in the following, the arguments of the coefficient functions are not displayed to simplify the notation.

**Theorem 3.1.** Let (A1)-(A3) be satisfied. Then, there exists a unique solution \((q, c) \in L^2(J; H(\text{div}, \Omega)) \times H^1(J; L^2(\Omega)) \cap L^2(J; H^2(\Omega))\) of Problem 1 with

\[
q = -D\nabla c + Qc.
\]

**Proof.** Let us first prove existence of a solution. Under the above assumptions, equation (1.1a) is equivalent to the advection-diffusion-reaction problem

\[
\partial_tc - \nabla \cdot (D\nabla c) + Q \cdot \nabla c + (\nabla \cdot Q + R)c = f
\]

in divergence form, equipped with the same boundary and initial conditions. The existence of a unique solution of (3.4) along with (1.1b)-(1.1c) in the space \(H^1(J; L^2(\Omega)) \cap L^2(J; H^2(\Omega))\) is provided by [15, Theorem IV.9.1, p. 341f.]. Note that since \(c\) is a weak solution of (1.1a), the pair \((q, c)\) with \(q = -D\nabla c + Qc\) satisfies

\[
(\partial tc, \phi) + (Rc, \phi) - (f, \phi) = (q, \nabla \phi)
\]

for all \(\phi \in H^1(\Omega)\) and a.e. \(t \in J\). Consequently, \(q \in L^2(J; H(\text{div}, \Omega))\) and \(\partial tc + \nabla \cdot q + Rc = f\) in \(L^2(\Omega)\) for a.e. \(t \in J\). This implies (3.2), whereas (3.1) follows from Green’s formula. To show uniqueness, assume that Problem 1 has two solutions \((q_1, c_1)\) and \((q_2, c_2)\) in \(L^2(J; H(\text{div}, \Omega)) \times H^1(J; L^2(\Omega))\). Defining \(c = c_1 - c_2\) and \(q = q_1 - q_2\), the error equations

\[
(D^{-1}q(t), v) - (\nabla \cdot v, c(t)) - (D^{-1}Qc(t), v) = 0,
\]

\[
(\partial tc(t), w) + (\nabla \cdot q(t), w) + (Rc(t), w) = 0
\]

hold for all \((v, w) \in H(\text{div}, \Omega) \times L^2(\Omega)\) and a.e. \(t \in J\). Next, we take \(v = q(t)\) and \(w = c(t)\) in (3.5)-(3.6) and add the resulting equations to find

\[
(D^{-1}q(t), q(t)) + (\partial tc(t), c(t)) + (Rc(t), c(t)) - (D^{-1}Qc(t), q(t)) = 0.
\]

Note that since \(c(0) = 0\), integrating the last equation from 0 to some \(t' < T\), using the ellipticity of \(D^{-1}\) and the uniform boundedness of \(D^{-1}, Q\) and \(R\), respectively, yields

\[
\int_0^{t'} \|q(t)\|^2_0 dt + \|c(t')\|^2_0 \leq C(1 + \frac{1}{4\delta}) \int_0^{t'} \|c(t)\|^2_0 dt + C\delta \int_0^{t'} \|q(t)\|^2_0 dt
\]

for any \(\delta > 0\). Finally, pushing back the last term for \(\delta\) sufficiently small and applying the Gronwall Lemma, we obtain \(\|c(t')\|_0 = 0\) and hence the uniqueness of the scalar variable. The uniqueness of the vector variable follows directly from \(\int_0^{t'} \|q(t)\|^2_0 dt = 0\) for all \(t' < T\). \(\square\)

### 3.2. The mixed finite element schemes.

In this section, we state the definition of the upwind-mixed hybrid method as defined in [20]. First, let us recall the following classical mixed approximation scheme for the conservation form problem, cf. [12, 13].
Problem 2. Let \( n \in \{1, \ldots, N\} \) and \( c_h^n \in W_h \) be given. Find \( (q_h^n, c_h^n) \in V_h \times W_h \) such that

\[
(D^{-1}q_h^n, v_h) - (\nabla \cdot v_h, c_h^n) - (D^{-1}Q^n c_h^n, v_h) = 0, \tag{3.7}
\]

\[
(\partial c_h^n, w_h) + (\nabla \cdot q_h^n, w_h) + (R^n c_h^n, w_h) = (f^n, w_h) \tag{3.8}
\]

for all \( (v_h, w_h) \in V_h \times W_h \).

It is well known that the linear algebraic system resulting when basis functions of \( V_h \) and \( W_h \) are employed in (3.7)-(3.8) is in general indefinite. Hence, common iterative solvers requiring a system matrix that is symmetric and positive definite may fail to converge. To overcome this problem, a hybridization process can be applied replacing \( V_h \) by the augmented space \( \tilde{V}_h \) and thus no longer requiring continuity of the normal fluxes across interelement boundaries, cf. [4]. The continuity constraints are then imposed with the help of Lagrange multipliers from the space \( \Lambda_h \) to ensure the equivalence of both formulations. Moreover, by applying a local elimination procedure, one can obtain a linear system for the Lagrange multipliers only, cf. Sec. 3.3. Hence, the resulting system has fewer unknowns than the one of the non-hybrid method. The mixed hybrid formulation associated with (3.7)-(3.8) reads as follows.

Problem 3. Let \( n \in \{1, \ldots, N\} \) and \( c_h^n \in W_h \) be given. Find \( (q_h^n, c_h^n, \lambda_h^n) \in \tilde{V}_h \times W_h \times \Lambda_h \) such that

\[
(D^{-1}q_h^n, v_h) - (\nabla \cdot v_h, c_h^n) - (D^{-1}Q^n c_h^n, v_h) = -\sum_{K \in T_h} \langle \lambda_h^n, v_h \cdot n \rangle_{\partial K}, \tag{3.9}
\]

\[
(\partial c_h^n, w_h) + (\nabla \cdot q_h^n, w_h) + (R^n c_h^n, w_h) = (f^n, w_h), \tag{3.10}
\]

\[
\sum_{K \in T_h} \langle \mu_h, q_h^n \cdot n \rangle_{\partial K} = 0 \tag{3.11}
\]

for all \( (v_h, w_h, \mu_h) \in \tilde{V}_h \times W_h \times \Lambda_h \).

The Lagrange multipliers may be regarded as an approximation of the scalar unknown on the element interfaces. In fact, they carry some extra information about the exact solution which can be used to reconstruct a higher order nonconforming approximation in \( L^2(\Omega) \). We refer to [2], where this was shown for an elliptic model problem. The use of the Lagrange multipliers in the discretization of the advective term is one of the key ideas of our upwind-mixed scheme, which reads as follows.

Problem 4. Define \( Q_h^n := \Pi_h Q^n \) having the representation

\[
Q_h^n = \sum_{K \in T_h} \sum_{E \in E(K)} Q_{KE}^n v_{KE}
\]

in the basis \( \{v_{KE}\}_{K \in T_h, E \in E(K)} \) of \( \tilde{V}_h \). Moreover, let \( n \in \{1, \ldots, N\} \) and \( c_h^n \in W_h \) be given. Find \( (q_h^n, c_h^n, \lambda_h^n) \in \tilde{V}_h \times W_h \times \Lambda_h \) with

\[
q_h^n = \sum_{K \in T_h} \sum_{E \in E(K)} q_{KE}^n v_{KE}, \quad c_h^n = \sum_{K \in T_h} c_{KE}^n \chi_K, \quad \lambda_h^n = \sum_{E \in E_h} \lambda_{KE}^n \mu_E
\]
Fig. 3.1: Scalar unknowns and Lagrange multiplier associated with the common edge of two adjacent triangles $K_1$ and $K_2$

satisfying

\[
(D^{-1}q_h^n, \nu_h) - (\nabla \cdot \nu_h, c_h^n) - \sum_{K \in T_h} \sum_{E \in \mathcal{E}(K)} Q_{KE}^n \alpha_{KE}^n(c^n_{KE}, \lambda_E^n)(D^{-1}e_{KE}, v_h) = - \sum_{K \in T_h} \langle \lambda_h^n, \nu_h \cdot n \rangle_{\partial K},
\]

\[(3.12)\]

\[
(\partial c^n_h, w_h) + (\nabla \cdot q^n_h, w_h) + (R^n c^n_h, w_h) = (f^n, w_h),
\]

\[(3.13)\]

\[
\sum_{K \in T_h} \langle \mu_h, q^n_h \cdot n \rangle_{\partial K} = 0
\]

\[(3.14)\]

for all $(\nu_h, w_h, \mu_h) \in \tilde{V}_h \times W_h \times \Lambda_h$, where the upwind weights are defined as

\[
\alpha_{KE}^n(c^n_{K}, \lambda_E^n) = \begin{cases} 
  c_K^n & \text{if } Q_{KE}^n \geq 0, \\
  2\lambda_E^n - c_K^n & \text{otherwise}.
\end{cases}
\]

The upwind weighting formula is motivated by the fact that the Lagrange multipliers provide an approximation of the scalar unknown on the edges. Then, heuristically, for two adjacent triangles $K_1$ and $K_2$ with the common edge $E$ (cf. Fig. 3.1), the value $\lambda_E^n$ provides an approximation of the mean value $\frac{1}{2}(c_{K_1}^n + c_{K_2}^n)$, or, in other words, $2\lambda_E^n - c_{K_1}^n$ may be assumed as an approximation for the value $c_{K_2}^n$ on the adjacent cell.

**Remark 1.** In coupled flow and transport simulations, the velocity field is often computed numerically, e.g. by solving Richards’ equation. If this is done using Raviart-Thomas elements of lowest order, the coefficients $Q_{KE}^n$ are directly available without evaluating the projection $\Pi_h$. For our error analysis, however, we continue to use $Q_h^n = \Pi_h Q^n$ in the definition of the scheme.

**Remark 2.** An extension of the method to three space dimensions using Raviart-Thomas elements of lowest order on tetrahedral grids is immediate, requiring only minor changes in the proof of convergence below, cf. Theorem 4.4.

For completeness, we state also the standard upwind-mixed method [8]. Since it approximates the diffusive flux $\tilde{q} = -D\nabla c$, the flux variable is denoted with a tilde.

**Problem 5.** Let $n \in \{1, \ldots, N\}$ and $c_h^{n-1}$ be given. Find $(\tilde{q}_h^n, c^n_h) \in V_h \times W_h$
satisfying
\[
(D^{-1}q^n_h, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, c^n_h) = 0 ,
\]
\[
(\partial_c c^n_h, w_h) + \sum_{K \in T_h} \sum_{E \in \mathcal{E}(K)} Q^n_{KE} \sigma^n_{KE} w_K + (R^n c^n_h, w_h) = (f^n, w_h)
\]
for all \((\mathbf{v}_h, w_h) \in V_h \times W_h, \text{ where } w_K = w_h|_K \) and
\[
\sigma^n_{KE} = \begin{cases} 
    c^n_K & \text{if } Q^n_{KE} \geq 0 , \\
    c^n_{K'} & \text{otherwise}
\end{cases}
\]
if \(E\) is a common edge of \(K\) and \(K'\) and
\[
\sigma^n_{KE} = \begin{cases} 
    c^n_K & \text{if } Q^n_{KE} \geq 0 , \\
    0 & \text{otherwise}
\end{cases}
\]
if \(E\) is a boundary edge.

### 3.3. Linear algebraic system of the upwind-mixed hybrid scheme

Let us assume for a moment that there exists a unique solution \((q^n_h, c^n_h, \lambda^n_h) \in V_h \times W_h \times \Lambda_h\) of Problem 4 having the basis representations
\[
q^n_h = \sum_{K \in T_h} \sum_{E \in \mathcal{E}(K)} q^n_{KE} \mathbf{v}_{KE} , \quad c^n_h = \sum_{K \in T_h} c^n_K \chi_K , \quad \lambda^n_h = \sum_{E \in \mathcal{E}_h} \lambda^n_E \mu_E .
\]
Employing these representations in the equations for the fluxes (3.12), using the basis functions as test functions in (3.12)-(3.14) and the relations
\[
(\nabla \cdot \mathbf{v}_{KE}, 1) = 1 , \quad (\mu_E, \mathbf{v}_{KE} \cdot \mathbf{n})_{\partial K} = \delta_{EE'} \quad \text{for } K \in T_h , \quad E, E' \in \mathcal{E}(K) ,
\]
the following system of linear equations for the fluxes is obtained on each element \(K\):
\[
\sum_{E \in \mathcal{E}(K)} q^n_{KE} (D^{-1} \mathbf{v}_{KE}, \mathbf{v}_{KE'}) - \sum_{E \in \mathcal{E}(K)} Q^n_{KE} \alpha^n_{KE} (c^n_K, \lambda^n_E) (D^{-1} \mathbf{v}_{KE}, \mathbf{v}_{KE'}) = c^n_K - \lambda_{E'} \quad \text{for all } K \in T_h , \quad E' \in \mathcal{E}(K) .
\]
Defining \(B_{KE} : = (D^{-1} \mathbf{v}_{KE}, \mathbf{v}_{KE'}) , \quad B_K : = \{B_{KE} \}_{E \in \mathcal{E}(K)} , \quad C : = (1, 1, 1)^T, \quad q^n_K = \{q^n_{KE}\}_{E \in \mathcal{E}(K)} , \quad \lambda^n_K = \{\lambda^n_E\}_{E \in \mathcal{E}(K)}, \) these equations can be rewritten in matrix form as
\[
B_K q^n_K = C c^n_K + B_K \begin{pmatrix} Q^n_{KE_1} \alpha^n_{KE_1} (c^n_K, \lambda^n_{E_1}) \\ Q^n_{KE_2} \alpha^n_{KE_2} (c^n_K, \lambda^n_{E_2}) \\ Q^n_{KE_3} \alpha^n_{KE_3} (c^n_K, \lambda^n_{E_3}) \end{pmatrix} - \lambda^n_K , \quad K \in T_h .
\]

It is easy to show that the matrices \(B_K\) are positive definite. Hence, the equations (3.15) can be solved explicitly for the flux unknowns on each element \(K \in T_h\). In the next step, the fluxes are inserted into the mass conservation equations (3.13) to obtain a representation \(c^n_K = c^n_K(\lambda^n_K)\) of the scalar unknown in terms of the Lagrange multipliers. Finally, by substituting the flux and scalar unknowns into the equations (3.14), a linear system for the Lagrange multipliers remains to be solved in each time step. Once it has been solved, the other variables can be reconstructed efficiently on each element. Note that this elimination process can only be applied if the upwind weights in (3.15) depend on variables on a single element \(K\).
4. Analysis of the upwind-mixed hybrid scheme. In this section, we present
the error analysis of the upwind-mixed hybrid method. First, let us demonstrate ex-
istence and uniqueness for the scheme (3.12)-(3.14).

Theorem 4.1. Assume (A1)-(A2). Then, for every $n \in \{1, \ldots, N\}$, Problem 4 has a unique solution.

Proof. Since the problem is linear, it suffices to show uniqueness. Thus, let $c_{h,1}^{n-1}$ be
given and assume that there exist two solutions $(q_{h,1}^{n}, c_{h,1}^{n}, \lambda_{h,1}^{n})$ and $(q_{h,2}^{n}, c_{h,2}^{n}, \lambda_{h,2}^{n}) \in V_{h} \times W_{h} \times \Lambda_{h}$ of (3.12)-(3.14) with the basis representations

$$c_{h,i}^{n} = \sum_{K \in T_{h}} c_{K,i}^{n} \chi_{K}, \quad \lambda_{h,i}^{n} = \sum_{E \in e_{h}^{i}} \lambda_{E,i}^{n} \mu_{E}, \quad i = 1, 2.$$  

Then, by subtracting, the error equations

$$ (D^{-1}(q_{h,1}^{n} - q_{h,2}^{n}), v_{h}) - (\nabla \cdot v_{h}, c_{h,1}^{n} - c_{h,2}^{n}) + \sum_{K \in T_{h}} \langle \lambda_{h,1}^{n} - \lambda_{h,2}^{n}, v_{h} \cdot n \rangle_{\partial K}$$

$$- \sum_{K \in T_{h}, E \in E(K)} Q_{K,E}(c_{K,1}^{n} - c_{K,2}^{n}, \lambda_{E,1}^{n} - \lambda_{E,2}^{n})(D^{-1}v_{K,E}, v_{h}) = 0,$$

$$\tau(\nabla \cdot (q_{h,1}^{n} - q_{h,2}^{n}), w_{h}) + \tau(R(c_{h,1}^{n} - c_{h,2}^{n}), w_{h}) = 0,$$

$$\sum_{K \in T_{h}} \langle \mu_{h}, (q_{h,1}^{n} - q_{h,2}^{n}) \cdot n \rangle_{\partial K} = 0$$

hold for all $(v_{h}, w_{h}, \mu_{h}) \in V_{h} \times W_{h} \times \Lambda_{h}$. Next, we proceed similarly as in [2] to obtain an estimate for the errors $\|\lambda_{h,1}^{n} - \lambda_{h,2}^{n}\|_{0,E}$ on the edges. For $K \in T_{h}$ and $E \in E(K)$ let $\tau_{E}$ denote the unique element of $V_{h}$ with supp($\tau_{E}$) $\subseteq K$ and

$$\tau_{E} \cdot n_{E'} = \begin{cases} \lambda_{h,1}^{n} - \lambda_{h,2}^{n} & \text{on } E = E', \\ 0 & \text{otherwise}. \end{cases}$$

Then, it follows from a simple scaling argument that

$$h_{K}||\tau_{E}||_{1} + ||\tau_{E}||_{0,K} \leq C(h_{K}^{1/2}||\lambda_{h,1}^{n} - \lambda_{h,2}^{n}||_{0,E},$$

and we obtain

$$\|\lambda_{h,1}^{n} - \lambda_{h,2}^{n}\|_{0,E} \leq C(h_{K}^{1/2}||q_{h,1}^{n} - q_{h,2}^{n}||_{0,K} + h_{K}^{-1/2}||c_{h,1}^{n} - c_{h,2}^{n}||_{0,K},$$

$$+ h_{K}^{1/2} \sum_{E' \in E(K)} |E'|(\|c_{h,1}^{n} - c_{h,2}^{n}\|_{0,K} + |\lambda_{h,1}^{n} - \lambda_{h,2}^{n}|_{0,E'})$$

$$\leq C(h_{K}^{1/2}||q_{h,1}^{n} - q_{h,2}^{n}||_{0,K} + h_{K}^{-1/2}||c_{h,1}^{n} - c_{h,2}^{n}||_{0,K},$$

$$+ h_{K} \sum_{E' \in E(K)} |\lambda_{h,1}^{n} - \lambda_{h,2}^{n}|_{0,E'}$$

by using $\tau_{E}$ as a test function in (4.1) and employing (4.11) and (4.4), respectively. Summing this estimate over the edges $E \in E(K)$ and pushing back the last term for $h$ sufficiently small yields

$$\|\lambda_{h,1}^{n} - \lambda_{h,2}^{n}\|_{0,E} \leq C(h_{K}^{1/2}||q_{h,1}^{n} - q_{h,2}^{n}||_{0,K} + h_{K}^{-1/2}||c_{h,1}^{n} - c_{h,2}^{n}||_{0,K}).$$
Next, we take $\mathbf{v}_h = \tau(q^n_{h,1} - q^n_{h,2})$, $w_h = c^n_{h,1} - c^n_{h,2}$ and $\mu_h = \tau(\lambda^n_{h,1} - \lambda^n_{h,2})$ as test functions in (4.1)-(4.3) and add the resulting equations to find

$$
\tau(D^{-1}(q^n_{h,1} - q^n_{h,2}), q^n_{h,1} - q^n_{h,2}) + \|c^n_{h,1} - c^n_{h,2}\|_0^2 + \tau(R(c^n_{h,1} - c^n_{h,2}), c^n_{h,1} - c^n_{h,2})
- \tau \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} Q^n_{KE} q^n_{KE}(\chi^n_{K,1} - \chi^n_{K,2}, \lambda^n_{E,1} - \lambda^n_{E,2})(D^{-1} \mathbf{v}_E, q^n_{h,1} - q^n_{h,2}) = 0 .
$$

Further, using (4.5), (2.3), the uniform ellipticity of $D^{-1}$ and the uniform boundedness of $Q$ and $R$, respectively, we obtain

$$
\|c^n_{h,1} - c^n_{h,2}\|_0^2 + \tau \|q^n_{h,1} - q^n_{h,2}\|_0^2 \leq C\tau \|c^n_{h,1} - c^n_{h,2}\|_0^2
+ C\tau \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} |E| (|\lambda^n_{E,1} - \lambda^n_{E,2}| + |c^n_{K,1} - c^n_{K,2}|) \|q^n_{h,1} - q^n_{h,2}\|_{0,K}
$$

$$
\leq C\tau \|c^n_{h,1} - c^n_{h,2}\|_0^2 + C\tau \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} \|\lambda^n_{h,1} - \lambda^n_{h,2}\|_{0,E} \|q^n_{h,1} - q^n_{h,2}\|_{0,K}
$$

$$
\leq C\tau \|c^n_{h,1} - c^n_{h,2}\|_0^2 + C\tau \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} \|\lambda^n_{h,1} - \lambda^n_{h,2}\|_{0,E} \|q^n_{h,1} - q^n_{h,2}\|_{0,K}
+ C\tau \sum_{K \in \mathcal{T}_h} h_K \|q^n_{h,1} - q^n_{h,2}\|_{0,K}
$$

$$
\leq C\tau \|c^n_{h,1} - c^n_{h,2}\|_0^2 + C\|c^n_{h,1} - c^n_{h,2}\|_0^2 + C\tau^2 \frac{1}{\delta} \|q^n_{h,1} - q^n_{h,2}\|_0^2
+ C\tau h \|q^n_{h,1} - q^n_{h,2}\|_0^2
$$

so that, by taking $\delta, \tau$ and $h$ sufficiently small, $c^n_{h,1} - c^n_{h,2}$ and $q^n_{h,1} - q^n_{h,2}$ vanish. Finally, $\lambda^n_{h,1} - \lambda^n_{h,2}$ vanishes according to (4.5). \(\square\)

Let us now turn to the error analysis of the fully discrete upwind-mixed scheme (3.12)-(3.14). One of the main steps in the proof of convergence is to obtain a priori estimates for the differences $|\lambda^n_E - c^n_E|$ which occur in the error equations obtained from (3.1)-(3.2) and (3.12)-(3.14). This is accomplished in the following Lemma by employing similar techniques as in the a posteriori error analysis in [2], where the Lagrange multipliers were used to obtain a higher order approximation of the scalar unknown by postprocessing.

**Lemma 4.2.** Let (A1)-(A3) hold and assume that the solution $(q, c)$ of Problem 1 satisfies (A4). Further, let $(q^n_{h,1}, c^n_{h,1}, \lambda^n_{h,1}) \in \bar{V}_h \times W_h \times \Lambda_h$ be the solution of Problem 4. Then, for $h$ sufficiently small, there exists a constant $C > 0$, independent of $n$ and $h$, such that

$$
\sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} |E|^2 (\lambda^n_E - c^n_E)^2 \leq C (\|c^n - c^n_h\|_0^2 + h^2 \|q^n - q^n_h\|_0^2 + h^2 \|\mathbf{c}_n\|_1^2 + h^2 \|\mathbf{c}_n\|_1^2) .
$$

**Proof.** In the first step, we show that

$$
\|\lambda^n_h - Q^n_{h,c} \|_{0,E} \leq C(\|h^{1/2} |q^n - q^n_h\|_{0,K} + h^{1/2} \sum_{E \in \mathcal{E}(K)} |E| |\lambda^n_E - c^n_E|
+ h^{1/2} \|Q^n_{h,c} - Q^n_{h,c} \|_{0,K} + h^{-1/2} \|c^n - c^n_h\|_{0,K}) ,
$$

(4.7)
where \( Q_h^0 \) denotes the \( L^2 \)-orthogonal projection onto \( L^2(\mathcal{E}_h) \), defined by
\[
(Q_h^0 c^n - c^n, 1)_E = 0 \quad \text{for all } E \in \mathcal{E}_h.
\]

Let now \( K \in \mathcal{T}_h \) and \( E \in \mathcal{E}(K) \) and define \( \tau_E \) as the unique element of \( V_h \) with \( \text{supp}(\tau_E) \subseteq K \) and
\[
\tau_E \cdot n_{E'} = \begin{cases} 
\lambda^n_h - Q_h^0 c^n & \text{on } E = E', \\
0 & \text{otherwise}.
\end{cases}
\]

A simple scaling argument shows that
\[
(4.8) \quad h_K \| \tau_E \|_1 + \| \tau_E \|_0 \leq C h_K^{1/2} \| \lambda^n_h - Q_h^0 c^n \|_{0,E}.
\]

Next, we use \( \tau_E \) as a test function in (3.12) to obtain
\[
(4.9) \quad (D^{-1} q_h^n, \tau_E) - (\nabla \cdot \tau_E, c^n) - \sum_{E' \in \mathcal{E}(K)} Q_{KE'}^n c^n c_{E'} \alpha_{KE'}^n (D^{-1} v_{KE'}, \tau_E) = - (\lambda^n_h, \lambda^n_h - Q_h^0 c^n)_E.
\]

Moreover, (3.3) and Green’s formula imply
\[
(4.10) \quad (D^{-1} q_h^n, \tau_E) - (\nabla \cdot \tau_E, c^n) - (D^{-1} Q_h^n c^n, \tau_E) = - (c^n, \lambda^n_h - Q_h^0 c^n)_E.
\]

Subtracting (4.9) from (4.10), using the definition of \( Q_h^0 \), the estimate
\[
(4.11) \quad |Q_{KE}^n| = |(Q^n \cdot n_E, 1)_E| \leq C |E|, \quad \text{the Cauchy-Schwarz inequality, (2.3) and (A1)}
\]
we find
\[
\| \lambda^n_h - Q_h^0 c^n \|^2_{0,E} = (\lambda^n_h - Q_h^0 c^n, \lambda^n_h - c^n)_E \\
= (D^{-1}(q^n - q_h^n), \tau_E) - (\nabla \cdot \tau_E, c^n - c_h^n) \\
- \sum_{E' \in \mathcal{E}(K)} Q_{KE'}^n c^n c_{E'} (D^{-1} v_{KE'}, \tau_E) + (D^{-1}(Q_h^0 c_h^n - Q^n c^n), \tau_E) \\
\leq C(\|q^n - q_h^n\|_{0,K} \|\tau_E\|_{0,K} + \|c^n - c_h^n\|_{0,K} \|\tau_E\|_{1,K}) \\
+ \sum_{E' \in \mathcal{E}(K)} |E'| \|\lambda_{E'} c_h^n - c_{KE'}|^2 \|\tau_E\|_{0,K} + \|Q_{KE}^n c_h^n - Q^n c_h^n\|_{0,K} \|\tau_E\|_{0,K}).
\]

Then, (4.7) follows from combining the last estimate with (4.8). Next, we use the Lagrange multipliers to define a nonconforming approximation \( \tilde{c}_h^n \) of \( c^n \) in the space \( P_1^{CR}(\mathcal{T}_h) \) of linear Crouzeix-Raviart elements by means of
\[
(4.12) \quad Q_h^0 c_h^n = \lambda_h^n.
\]

Note that \( \tilde{c}_h^n \) is uniquely determined according to [2, Lemma 2.1], see also the remark on boundary elements having a curved edge after Lemma 4.2 in [3]. Similarly, we define \( \tilde{c}_h^n \in P_1^{CR}(\mathcal{T}_h) \) to be the nonconforming projection of \( c^n \) by means of
\[
(4.13) \quad Q_h^0 (c^n - \tilde{c}_h^n) = 0.
\]
Then, by standard arguments, it follows that

\begin{equation}
||\tilde{c}_h^n - c^n||_0 \leq C h ||c^n||_1 .
\end{equation}

Moreover, from (4.12) and (4.13), we have

\[ Q^n_h(\tilde{c}_h^n - \tilde{c}_h^n) = \lambda^n_E - Q^n_h c^n , \]

which implies

\[ ||\tilde{c}_h^n - c^n||_0, K \leq C h^{1/2} \sum_{E \in \mathcal{E}(K)} \lambda^n_h - Q^n_h c^n||_0, E \]

by [2, Lemma 2.1]. Combining this with (4.7), we obtain

\[ ||\tilde{c}_h^n - c^n||_0, K \leq C (h K ||q^n - q^n_h||_0, K + ||c^n - c^n_h||_0, K + h K ||Q^n_h c^n - Q^n c^n||_0, K \]

\[ + h K \sum_{E \in \mathcal{E}(K)} |E||\lambda^n_E - c^n_K| \]

so that, by summing over all elements,

\[ ||\tilde{c}_h^n - c^n||^2_0 \leq C (h^2 ||q^n - q^n_h||^2_0 + ||c^n - c^n_h||^2_0 + h^2 ||Q^n_h c^n - Q^n c^n||^2_0 \]

\[ + h^2 \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} |E|^2 (\lambda^n_E - c^n_K)^2 \]

(4.15)

Hence, from (4.14) and (4.15), we get

\[ ||\tilde{c}_h^n - c^n||^2_0 \leq 3 ||c^n_h - c^n||^2 + 3 ||c^n - c^n_h||^2 + 3 ||c^n - c^n_h||^2_0 + h^2 ||Q^n_h c^n - Q^n c^n||^2_0 \]

\[ + h^2 \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} |E|^2 (\lambda^n_E - c^n_K)^2 \]

(4.16)

Now, we use that for any piecewise linear polynomial \( p_h \in \mathcal{P}_1(\mathcal{T}_h) \), the estimate

\[ ||p_h||_{L^1(\partial K)} \leq C ||p_h||_0, K \]

holds, cf. [4, p. 112]. Then, for any \( K \in \mathcal{T}_h \) and \( E \in \mathcal{E}(K) \),

\[ |E|(\lambda^n_E - c^n_K) = (\tilde{c}_h^n - c^n_h, 1)_E \leq C ||\tilde{c}_h^n - c^n_h||_{L^1(\partial K)} \leq C ||\tilde{c}_h^n - c^n_h||_0, K \]

Finally, combining the last estimate with (4.16) yields

\[ \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} |E|^2 (\lambda^n_E - c^n_K)^2 \leq C ||\tilde{c}_h^n - c^n_h||^2_0 \leq C (h^2 ||q^n - q^n_h||^2_0 + ||c^n - c^n_h||^2_0 \]

\[ + h^2 ||Q^n_h c^n - Q^n c^n||^2_0 + h^2 \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} |E|^2 (\lambda^n_E - c^n_K)^2 + h^2 ||c^n||^2_1) \]

and (4.6) follows by pushing back the last term on the right hand side for sufficiently small \( h \). \( \square \)

For later use, we state the following discrete integration by parts formula.
Lemma 4.3. Let \((a_n)_{n \in \mathbb{N}_0}\) and \((b_n)_{n \in \mathbb{N}_0}\) be real sequences. Then, for any \(m \in \mathbb{N}\),

\[
\sum_{n=1}^{m} (a_n - a_{n-1}) b_n = a_m b_m - a_0 b_0 - \sum_{n=1}^{m} a_{n-1} (b_n - b_{n-1}).
\]

Theorem 4.4. Assume (A1)-(A3) and let \((q, c)\) denote the unique solution of Problem 1. Moreover, let \(\tau\) and \(h\) be sufficiently small such that for every \(n \in \{1, \ldots, N\}\) there exists a unique solution \((q^n_h, c^n_h, \lambda^n_h) \in V_h \times W_h \times \Lambda_h\) of Problem 4. Then, if (A4) is satisfied and \(c_h = P_h c_0\), there exists a constant \(C > 0\), independent of \(\tau\) and \(h\), such that

\[
(4.17) \quad \max_{m \in \{1, \ldots, N\}} \|c(t^n) - c^n_h\|_0^2 + \tau \sum_{n=1}^{N} \|q(t^n) - q^n_h\|_0^2 \leq C(\tau^2 + h^2).
\]

Proof. We take \(t = t^n\) in (3.1)-(3.2), subtract (3.12)-(3.13) from the resulting equations and use the projector properties (2.1a)-(2.1b) to obtain the error equations

\[
(4.18) \quad (D^{-1}(q^n - q^n_h), v_h) = (\nabla \cdot v_h, P_h c^n - c^n_h) - (D^{-1}Q^n c^n, v_h)
\]

\[+ \sum_{K \in T_h} \sum_{E \in \mathcal{E}(K)} Q^n_{KE} \alpha^n_{KE}(D^{-1}v_{KE}, v_h) = - \sum_{K \in T_h} \langle \lambda^n_{h}, v_h \cdot n \rangle_{\partial K},\]

\[
(4.19) \quad (\partial_t c^n - \bar{\partial} c^n_h, w_h) + (\nabla \cdot (P_h q^n - q^n_h), w_h) + (R^n(c^n - c^n_h), w_h) = 0,
\]

\[
(4.20) \quad \sum_{K \in T_h} \langle (P_h q^n - q^n_h) \cdot n, \mu_h \rangle_{\partial K} = 0
\]

for all \((v_h, w_h, \mu_h) \in \dot{V}_h \times W_h \times \Lambda_h\). Note that our regularity assumptions allow the evaluation of \(c\) and \(q\) and all coefficient functions at \(t = t^n\). Taking \(v_h = \Pi_h q^n - q^n_h\), \(w_h = P_h c^n - c^n_h\) and \(\mu_h = \lambda^n_h\) as test functions in (4.18)-(4.20) and adding the resulting equations yields

\[
(4.21) \quad (D^{-1}(q^n - q^n_h), \Pi_h q^n - q^n_h) - (D^{-1}Q^n c^n, \Pi_h q^n - q^n_h)
\]

\[+ (\partial_t c^n - \bar{\partial} c^n_h, P_h c^n - c^n_h) + \sum_{K \in T_h} \sum_{E \in \mathcal{E}(K)} Q^n_{KE} \alpha^n_{KE}(D^{-1}v_{KE}, \Pi_h q^n - q^n_h)
\]

\[+ (R^n(c^n - c^n_h), P_h c^n - c^n_h) = 0.
\]

The last identity can be rewritten equivalently as

\[
(4.21) \quad (D^{-1}(q^n - q^n_h), \Pi_h q^n - q^n_h) + (D^{-1}(Q^n_{h} c^n - Q^n c^n), \Pi_h q^n - q^n_h)
\]

\[+ \sum_{K \in T_h} \sum_{E \in \mathcal{E}(K)} Q^n_{KE} \alpha^n_{KE}(D^{-1}v_{KE}, \Pi_h q^n - q^n_h)
\]

\[+ (\partial_t c^n - \bar{\partial} c^n_h, P_h c^n - c^n_h) + (R^n(c^n - c^n_h), P_h c^n - c^n_h) = 0.
\]

Further, for any \(m \in \mathbb{N}\) with \(1 \leq m \leq N\), by summing (4.21) from \(n = 1, \ldots, m\),
We now proceed to estimate separately each of the terms $(4.24)$ multiplying by $2\tau$ and expanding, we obtain

\[
T_1 + \ldots + T_9 := 2\tau \sum_{n=1}^{m} (D^{-1}(q^n - q^n_h), q^n - q^n_h) \\
+ 2\tau \sum_{n=1}^{m} (D^{-1}(q^n - q^n_h), \Pi_h q^n - q^n) \\
+ 2\tau \sum_{n=1}^{m} (D^{-1}(Q^n_q c^n_h - Q^n c^n_h), \Pi_h q^n - q^n_h) \\
+ 2\tau \sum_{n=1}^{m} (\partial_h c^n - \overline{\partial}c^n, c^n - c^n_h) \\
+ 2\tau \sum_{n=1}^{m} (\partial_h c^n - \overline{\partial}c^n, P_h c^n - c^n) \\
+ 2\tau \sum_{n=1}^{m} (\overline{\partial}(c^n - c^n_h), c^n - c^n_h) \\
+ 2\tau \sum_{n=1}^{m} (\overline{\partial}(c^n - c^n_h), P_h c^n - c^n) \\
+ 2\tau \sum_{n=1}^{m} (R^n(c^n - c^n_h), P_h c^n - c^n_h) \\
+ 2\tau \sum_{n=1}^{m} \sum_{K \in T_h} \sum_{E \in E(K)} Q^n_{KE}(\alpha^n_{KE} - c^n_h)(D^{-1}v_{KE}, \Pi_h q^n - q^n_h) \\
= 0.
\]

We now proceed to estimate separately each of the terms $T_1, \ldots, T_9$. First, the terms $T_1$ and $T_6$ are bounded from below. Note that since $D$ is uniformly positive definite, $D^{-1}$ is also uniformly positive definite. Thus we have

\[
(4.22) \quad T_1 = 2\tau \sum_{n=1}^{m} (D^{-1}(q^n - q^n_h), q^n - q^n_h) \geq C\tau \sum_{n=1}^{m} \|q^n - q^n_h\|_0^2.
\]

By the identity $2a(a - b) = a^2 - b^2 + (a - b)^2$, we find

\[
T_6 = 2 \sum_{n=1}^{m} ((c^n - c^n_h) - (c^{n-1} - c^{n-1}_h), c^n - c^n_h) \\
= \sum_{n=1}^{m} \|c^n - c^n_h\|_0^2 - \sum_{n=1}^{m} \|c^{n-1} - c^{n-1}_h\|_0^2 + \sum_{n=1}^{m} \|c^n - c^n_h - c^{n-1} + c^{n-1}_h\|_0^2 \\
\geq \|c^n - c^n_h\|_0^2 - \|c^n_h\|_0^2.
\]

All other terms are passed to the right hand side and bounded from above. Using a weighted Young’s inequality, we get for any $\delta_2 > 0$

\[
(4.24) \quad |T_2| \leq C\delta_2 \tau \sum_{n=1}^{m} \|q^n - q^n_h\|_0^2 + C\tau \sum_{n=1}^{m} \|\Pi_h q^n - q^n\|_0^2.
\]
The term $T_3$ is rewritten as

$$T_3 = 2\tau \sum_{n=1}^{m} (D^{-1}(Q_h^n c_h^n - Q^n c^n), q^n - q_h^n)$$

$$+ 2\tau \sum_{n=1}^{m} (D^{-1}(Q_h^n c_h^n - Q^n c^n, \Pi_h q^n - q^n) =: T_{31} + T_{32}.$$ 

Next, we take $h$ sufficiently small and use the $L^\infty(\Omega)$ approximation property

$$\|Q_h^n - Q^n\|_{L^\infty(\Omega)} = \|\Pi_h Q^n - Q^n\|_{L^\infty(\Omega)} \leq C h \|Q^n\|_{W^{1,\infty}(\Omega)}$$

of $\Pi_h$ [14] and the regularity of $Q$ to find

$$\|Q_h^n c_h^n - Q^n c^n\|_0 \leq \|Q_h^n (c^n - c_h^n)\|_0 + \|Q^n - Q_h^n\| c^n_0$$

$$\leq \|Q_h^n\|_{L^\infty(\Omega)} \|c^n - c_h^n\|_0 + \|Q^n - Q_h^n\|_{L^\infty(\Omega)} c^n_0$$

$$\leq C (\|c^n - c_h^n\|_0 + h \|c^n\|_0).$$

Then, it follows from the uniform boundedness of $D^{-1}$ and a weighted Young’s inequality that for any $\delta_3 > 0$

$$|T_{31}| \leq \frac{C \tau}{\delta_3} \sum_{n=1}^{m} \|Q_h^n c_h^n - Q^n c^n\|_0^2 + C \delta_3 \tau \sum_{n=1}^{m} \|q^n - q_h^n\|_0^2$$

$$\leq \frac{C \tau}{\delta_3} \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2 + \frac{C h^2 \tau}{\delta_3} \sum_{n=1}^{m} \|c^n\|_0^2 + C \delta_3 \tau \sum_{n=1}^{m} \|q^n - q_h^n\|_0^2.$$ 

Similarly, we derive the estimate

$$|T_{32}| \leq C \tau \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2 + C h^2 \tau \sum_{n=1}^{m} \|c^n\|_0^2 + C \tau \sum_{n=1}^{m} \|\Pi_h q^n - q^n\|_0^2.$$ 

Next, we use Bochner’s inequality, the Cauchy-Schwarz inequality and the regularity of $c$ to find

$$|T_4| \leq \frac{1}{\tau} \sum_{n=1}^{m} \|\tau \delta_t c^n - c^n + c^{n-1}\|_0^2 + \tau \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2$$

$$\leq \frac{1}{\tau} \sum_{n=1}^{m} \left\| \int_{t_{n-1}}^{t_n} \int_{\mathcal{A}} \delta_t c(\eta) \, d\eta \, ds \right\|_0^2 + \tau \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2$$

$$\leq \tau^2 \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \|\delta_t c(\eta)\|_0^2 \, d\eta + \tau \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2$$

$$\leq \tau^2 \|\delta_t c\|_{L^2(J; L^2(\Omega))}^2 + \tau \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2$$

$$\leq C \tau^2 + \tau \sum_{n=1}^{m} \|c^n - c_h^n\|_0^2.$$
Analogously,

\[ |T_5| \leq \tau^2 \| \partial_t u \|_{L^2(J;L^2(\Omega))}^2 + \tau \sum_{n=1}^{m} \| P_h c^n - c^n \|_0^2 \]

(4.29)

\[ \leq C \tau^2 + \tau \sum_{n=1}^{m} \| P_h c^n - c^n \|_0^2 . \]

For \( T_7 \), we employ Lemma 4.3 and the definition \( c_h^n = P_h c_0 \) to get

\[ |T_7| \leq \delta_7 \| c^m - c_h^m \|_0^2 + \frac{1}{\delta_7} \| P_h c^m - c^n \|_0^2 + (2 + \tau) \| P_h c_0 - c_0 \|_0^2 \]

(4.30)

\[ + \tau \sum_{n=1}^{m-1} \| c^n - c_h^n \|_0^2 + \frac{1}{\tau} \sum_{n=1}^{m} \| (P_h - I)(c^n - c^{n-1}) \|_0^2 , \]

where \( I \) denotes the identity operator. Denoting the last term on the right hand side of (4.30) by \( T_{71} \), we obtain from Bochner’s inequality, the Cauchy-Schwarz inequality and the regularity of \( c \) that

\[ T_{71} \leq \frac{Ch^2}{\tau} \sum_{n=1}^{m} \| c^n - c^{n-1} \|_1^2 \leq \frac{Ch^2}{\tau} \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n} \| \partial_t c(\eta) \|_{L^2(J;H^1(\Omega))}^2 \]

\[ \leq Ch^2 \int_{0}^{t^m} \| \partial_t c(\eta) \|_{L^2(J;H^1(\Omega))}^2 d\eta \leq Ch^2 \| \partial_t c \|_{L^2(J;H^1(\Omega))} \]

\[ \leq Ch^2 . \]

Consequently,

\[ |T_7| \leq \delta_7 \| c^m - c_h^m \|_0^2 + \frac{1}{\delta_7} \| P_h c^m - c^n \|_0^2 + (2 + \tau) \| P_h c_0 - c_0 \|_0^2 \]

(4.31)

\[ + \tau \sum_{n=1}^{m} \| c^n - c_h^n \|_0^2 + Ch^2 . \]

Further, from the uniform boundedness of \( R \), we get

\[ |T_8| \leq C \tau \sum_{n=1}^{m} \| c^n - c_h^n \|_0^2 + C \tau \sum_{n=1}^{m} \| P_h c^n - c^n \|_0^2 \]

(4.32)

\[ \leq C \tau \sum_{n=1}^{m} \| c^n - c_h^n \|_0^2 + C \tau \sum_{n=1}^{m} \| P_h c^n - c^n \|_0^2 . \]

For the remaining term \( T_9 \), we combine (4.11) and (2.3) to find

\[ |T_9| \leq \frac{C}{\delta_9} \sum_{n=1}^{m} \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{F}(K)} |E|^2 (\lambda_K^n - c_h^n)^2 + C \delta_9 \tau \sum_{n=1}^{m} \| \Pi_h q^n - q^n \|_0^2 \]

\[ + C \delta_9 \tau \sum_{n=1}^{m} \| q^n - q_h^n \|_0^2 , \]
and it follows from Lemma 4.2 and (4.25) that

\[(4.33)\]

\[\left| T_h \right| \leq \frac{C_{\delta_9}}{\delta_9} \left( h^2 \tau \sum_{n=1}^{m} \| c^n \|_0^2 + \tau \sum_{n=1}^{m} \| c^n - c_h^n \|_0^2 + h^2 \tau \sum_{n=1}^{m} \| q^n - q_h^n \|_0^2 + h^2 \tau \sum_{n=1}^{m} \| c^n - c_h^n \|_0^2 \right) + h^4 \tau \sum_{n=1}^{m} \| c^n \|_0^2 + C\tau \delta_9 \sum_{n=1}^{m} \| \Pi_h q^n - q^n \|_0^2 + C\tau \delta_9 \sum_{n=1}^{m} \| q^n - q_h^n \|_0^2 .\]

Finally, by collecting the estimates (4.22)-(4.33), choosing \( \delta_2, \delta_3, \delta_7, \delta_9, \tau \) and \( h \) sufficiently small and pushing back terms to the left hand side, we obtain

\[\| c^m - c_h^m \|_0^2 + \tau \sum_{n=1}^{m} \| q^n - q_h^n \|_0^2 \leq C\tau^2 + Ch^2 + C\tau \sum_{n=1}^{m} \| \Pi_h q^n - q^n \|_0^2 + C\tau \sum_{n=1}^{m} \| P_h c^n - c^n \|_0^2 + C\tau \sum_{n=1}^{m} \| P_h c_h^n - c_h^n \|_0^2 + C\tau \sum_{n=1}^{m} \| c^n - c_h^n \|_0^2 + Ch^2 \tau \sum_{n=1}^{m} \| c^n \|_0^2 \]

and (4.17) follows from the properties of the projectors, the regularity of \( q \) and \( c \) and by applying the discrete Gronwall Lemma. \( \square \)

Remark 3. The proof of Theorem 4.4 applies to all choices of the weights \( \alpha_{n}^{\nu E} \) satisfying

\[| \alpha_{n}^{\nu E}(c_K^n, \lambda_E^n) - c_K^n | \leq C | \lambda_E^n - c_K^n | .\]

Hence, the convergence estimate (4.17) continues to hold for the classical scheme (3.9)-(3.11) and for the partial upwind scheme with the weights defined by

\[\alpha_{n}^{\nu E}(c_K^n, \lambda_E^n) = \begin{cases} 
        e_K^n & \text{if } Q_{nE}^n \geq 0 , \\
        (1 - \nu_E)c_K^n + \nu_E(2\lambda_E^n - c_K^n) & \text{otherwise ,}
\end{cases} \]

where \( \nu_E \in [0, \frac{1}{2}] \) is a coefficient describing the amount of upstream weighting. The same applies to the method we obtain by taking

\[\alpha_{n}^{\nu E}(c_K^n, \lambda_E^n) = \lambda_E^n ,\]

which was also tested in [20] and proved to be robust for moderately advection-dominated problems in numerical experiments.

Remark 4. Note that since we applied the discrete Gronwall Lemma, the constant in the error estimate (4.17) is potentially large. This is in accordance with the convergence analysis of the upwind-mixed methods in [6] and [7]. In [9], the use of the Gronwall Lemma was avoided, but only suboptimal convergence was obtained for the semidiscrete problem. As indicated in Remark 3, our proof works for a whole class of methods including the classical one, so the error estimate cannot be expected to be independent of the Péclet number.

5. Numerical results. In this section we provide computational evidence that the error bound derived in the previous section is sharp and compare our upwind-mixed hybrid method to the upwind-mixed method of Dawson [8]. Further numerical
Table 5.1: Numerical results for the upwind-mixed hybrid method

<table>
<thead>
<tr>
<th>Ref. level</th>
<th># Unknowns</th>
<th>Error $E_{\tau,h}$</th>
<th>Reduction</th>
<th>CPU time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>3.685e-2</td>
<td>1.60</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>56</td>
<td>2.297e-2</td>
<td>1.85</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>208</td>
<td>1.236e-2</td>
<td>1.95</td>
<td>0.09</td>
</tr>
<tr>
<td>4</td>
<td>800</td>
<td>6.345e-3</td>
<td>1.98</td>
<td>0.47</td>
</tr>
<tr>
<td>5</td>
<td>3136</td>
<td>3.205e-3</td>
<td>1.99</td>
<td>3.88</td>
</tr>
<tr>
<td>6</td>
<td>12416</td>
<td>1.610e-3</td>
<td>2.00</td>
<td>33.11</td>
</tr>
<tr>
<td>7</td>
<td>49408</td>
<td>8.067e-4</td>
<td>2.00</td>
<td>305.90</td>
</tr>
</tbody>
</table>

Table 5.2: Numerical results for the upwind-mixed method [8]

<table>
<thead>
<tr>
<th>Ref. level</th>
<th># Unknowns</th>
<th>Error $E_{\tau,h}$</th>
<th>Reduction</th>
<th>CPU time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>3.507e-2</td>
<td>1.54</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>88</td>
<td>2.264e-2</td>
<td>1.83</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>226</td>
<td>1.234e-2</td>
<td>1.94</td>
<td>0.10</td>
</tr>
<tr>
<td>4</td>
<td>1312</td>
<td>6.349e-3</td>
<td>1.98</td>
<td>0.66</td>
</tr>
<tr>
<td>5</td>
<td>5184</td>
<td>3.208e-3</td>
<td>1.99</td>
<td>5.79</td>
</tr>
<tr>
<td>6</td>
<td>20608</td>
<td>1.611e-3</td>
<td>2.00</td>
<td>57.52</td>
</tr>
<tr>
<td>7</td>
<td>82176</td>
<td>8.068e-4</td>
<td>2.00</td>
<td>709.00</td>
</tr>
</tbody>
</table>

examples including tests with high Péclet numbers are presented in [20]. In our computations, we let $\Omega = (0,1)^2$, $J = (0,1)$ and solve the model problem

$$\partial_t c - \nabla \cdot (D \nabla c - Qc) = f \text{ in } J \times \Omega ,$$
$$c = c_0 \text{ on } \{0\} \times \Omega ,$$
$$c = 0 \text{ on } J \times \partial \Omega$$

with $D = 1$, $Q = (0, -1)^T$, $c_0(x,y) = x(1-x)y(1-y)$. The source term $f$ is chosen such that $c(t,x,y) = x(1-x)y(1-y)e^{-t}$ is the analytical solution of the problem. A simple uniform triangular mesh with $h = \sqrt{2}$ was used as a coarse grid and $\tau = 0.5$ was chosen as the initial time step size. In each level of refinement, $h$ and $\tau$ were halved and the errors

$$E_{\tau,h} = \left\{ \|c(t^N) - c_h^N\|_0^2 + \tau \sum_{n=1}^{N} \|q(t^n) - \tilde{q}_h^n\|_0^2 \right\}^{1/2}$$

were computed. For the upwind scheme of Dawson, we used $q_h^n = \tilde{q}_h^n + Qc_h^n \in RT_0(\Omega, T_h)$ as an approximation of the total flux, where $q_h$ denotes the flux variable of the scheme which approximates the diffusive flux $\hat{q} = -D\nabla c$. Both methods were implemented in the software package M++ [24, 25]. To handle the saddle point problems arising from the standard upwind-mixed scheme, the linear systems were solved in each time step with a direct solver based on the SuperLU library [11]. The results of the computations are summarized in Tables 5.1 and 5.2 and clearly indicate optimal first order convergence in $h$ and $\tau$ for our upwind-mixed hybrid scheme. Hence, the error bound of Theorem 4.4 is sharp. Moreover, the magnitude
of the errors almost coincides for both methods, while the computation time was 50% lower on the finest mesh with our hybrid method. On the one hand, this is due to a lower number of global unknowns resulting from hybridization and employing a local elimination process, cf. Sec. 3.3. On the other hand, the linear system associated with our method is sparser than the one corresponding to the non-hybrid method. Hence, the linear solver is faster and less memory is required during the computation.

6. Conclusion. In this work we analyzed an upwind-mixed hybrid finite element scheme for linear parabolic advection-diffusion-reaction problems. The method is based on the Raviart-Thomas mixed finite element of lowest order on triangular meshes and uses an upwind weighting formula for the discretization of the advective term. The definition of the upwind weights involves interelement multipliers which are introduced into the system by hybridization to enforce continuity of the normal fluxes across interelement boundaries. More precisely, the Lagrange multipliers are used to obtain an approximation of the scalar unknowns on adjacent cells to avoid that they are accessed directly. Consequently, the number of unknowns of the linear system can be reduced by eliminating variables locally, which makes our method more efficient than upwind-schemes using information of neighbour cells to define the upwind weights.

Existence and uniqueness of the fully discrete scheme were shown and optimal first order convergence in time and space was obtained by techniques adapted from the a posteriori error analysis for the Lagrange multipliers in [2]. Our error analysis also applies to a partial upwind scheme which can be used to reduce the amount of artificial diffusion, or if the Lagrange multipliers are used directly to discretize the advective term, cf. [20]. An extension of the scheme to tetrahedral meshes in three space dimensions is straightforward. In a computational experiment, the error bounds were confirmed numerically, and the method provided the same accuracy as the upwind mixed method of Dawson [8], while up to 50% of computation time could be saved by local elimination of unknowns and a sparser structure of the system matrix.

REFERENCES

2007/1  Cao, Y. / Eikemo, B. / Helmig, R.: Fractional flow formulation for two-phase flow in porous media

2008/1  Helmig, R. / Weiss, A. / Wohlmuth, B.: Variational inequalities for modeling flow in heterogeneous porous media with entry pressure

2008/2  Cao, Y. / Helmig, R. / Wohlmuth, B.: Convergence study and comparison of the multipoint flux approximation L-method

2008/3  van Duijn, C.J. / Pop, I.S. / Niessner, J. / Hassanizadeh, S.M.: Philip’s redistribution problem revisited: the role of fluid-fluid interfacial areas


2008/6  Cao, Y. / Helmig, R. / Wohlmuth, B.: Geometrical interpretation of the multipoint flux approximation L-method

2008/7  Vervoort, R.W. / van der Zee, S.E.A.T.M.: Simulating the effect of capillary flux on the soil water balance in a stochastic ecohydrological framework


2008/10 Wolff, M.: Comparison of mathematical and numerical models for two-phase flow in porous media


2008/12 Cao, Y. / Helmig, R. / Wohlmuth, B.: Convergence of the multipoint flux approximation L-method for homogeneous media on uniform grids

2008/13 Ochs, S.O.: Development of a multiphase multicomponent model for PEMFC

2008/14 Walter, L.: Towards a model concept for coupling porous gas diffusion layer and gas distributor in PEM fuel cells

2009/1  Hægland, H. / Assteerawatt, A. / Helmig, R. / Dahle, H.K.: Streamline approach for a discrete fracture-matrix system


2009/3  Heimann, F.: An unfitted discontinuous Galerkin method for two-phase flow

2009/4  Hilfer, R. / Doster, F.: Percolation as a basic concept for macroscopic capillarity
van Noorden, T.L. / Pop, I.S. / Ebigbo, A. / Helmig, R.: An effective model for biofilm growth in a thin strip

Baber, K.: Modeling the transfer of therapeutic agents from the vascular space to the tissue compartment (a continuum approach)

Faigle, B.: Two-phase flow modeling in porous media with kinetic interphase mass transfer processes in fractures

Fritz, J. / Flemisch, B. / Helmig, R.: Multiphysics modeling of advection-dominated two-phase compositional flow in porous media

Støverud, K.: Modeling convection-enhanced delivery into brain tissue using information from magnetic resonance imaging


Kissling, F. / Rohde, C.: The computation of non-classical shock waves with a heterogeneous multiscale method

Rosenbrand, E.: Modelling biofilm distribution and its effect on two-phase flow in porous media

Schöniger, A.: Parameter estimation by ensemble Kalman filters with transformed data


Linders, B.: Experimental investigations on horizontal redistribution

Rau, M.T.: Geostatistical analysis of three-dimensional hydraulic conductivity fields by means of maximum Gauss copula

Kraus, D.: Two phase flow in homogeneous porous media - The role of dynamic capillary pressure in modeling gravity driven fingering

Brugman, R.: Dimensionless analysis of convection enhanced drug delivery to brain tissues

Sinsbeck, M.: Adaptive grid refinement for two-phase flow in porous media

Kissling, F. / Helmig, R. / Rohde, C.: A multi-scale approach for the modelling of infiltration processes in the unsaturated zone

Köppl, T. / Wohlmuth, B. / Helmig, R.: Reduced one-dimensional modelling and numerical simulation for mass transport in fluids

Kumar, K. / Pop, I.S. / Radu, F.A.: Convergence analysis for a conformal discretization of a model for precipitation and dissolution in porous media

Hommel, J.: Modelling biofilm induced calcite precipitation and its effect on two phase flow in porous media
2012/4  Estrella, D.: Experimental and numerical approximation methods for zero-valent iron transport around injection wells

2012/5  Heimhuber, R.: Efficient history matching for reduced reservoir models with PCE-based bootstrap filters

2012/6  Kissling, F. / Karlsen, K.H.: On the singular limit of a two-phase flow equation with heterogeneity and dynamic capillary pressure

2012/7  Fritz, S.: Experimental investigations of water infiltration into unsaturated soil - Analysis of dynamic capillarity effects

2012/8  Strohmer, V.: Numerische Analysis von nahezu parallelen Strömungen in porösen Medien

2012/9  Kissling, F. / Rohde, C.: The computation of nonclassical shock waves in porous media with a heterogeneous multiscale method: The multidimensional case