Non-linearities and Upscaling in Porous Media

Compressible Multicomponent Flow in a Porous Medium: Maxwell–Stefan Diffusion

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List of Symbols

$\bar{G}$ Closure of set $G$

$\partial \Omega$ Boundary of set $\Omega$

$C([0,T],X)$ Space of continuous functions on $[0,T]$ with values in $X$

$C^s([0,T],X)$ Space of $s$-times continuous differentiable functions on $[0,T]$ with values in $X$

$C^s_b([0,T],X)$ Space of $s$-times continuous differentiable functions on $[0,T]$ with values in $X$ and bounded derivatives

$C^\infty(\Omega)$ Space of infinitely continuous differentiable functions on $\Omega$

$\partial^\alpha f$ Partial derivative of $f$ with respect to the multiindex $\alpha$

$\partial_t f$ Partial derivative of $f$ with respect to $t$

$F_U(U)$ Total derivative of function $F$

$\text{div} F$ Divergence of vector field $F$

$\nabla f$ Gradient of $f$

$\text{diag}(x)$ Diagonal matrix with entries of vector $x$ on the diagonal

$\subset \subset$ Compact embedding

$\lfloor \cdot \rfloor$ Floor function

$I_d$ Identity tensor of dimension $d$

$\prec, \preceq, \succ, \succeq$ Partial ordering for symmetric matrices

$L^2(\Omega)$ Space of square-integrable functions on $\Omega$

$\| \cdot \|_s$ Standard norm in $H^s$

$| \cdot |$ Norm of vector or matrix

$H^s$ Sobolev space of $L^2$-functions with distribution derivatives of order $\leq s \in \mathbb{N}_0$ in $L^2$

$\otimes$ Tensor product

$\mathbf{v}^T$ Transposed of vector or matrix $\mathbf{v}$
1 Introduction

Multicomponent flows in porous media appear in various fields of applications such as fuel cells, oxygen sensors and respiratory airways. As stated in [4], the bronchial tree can be divided into two parts. In the lower part the velocity of the air is very small, so the dynamic of a gas mixture is mainly dictated by diffusive effects. For treatment of certain diseases of the lung, a gas mixture is used to improve the patients well-being. Mathematical models can be used to analyse how to achieve the greatest benefit for the patient. In this and other situations the classical Fickian diffusion law is too simplistic. This law was introduced by Fick [11] and means basically that the flux goes from regions with high concentration to regions with lower concentration and the flux is proportional to the gradient of the concentration. Important occurring effects, for instance uphill diffusion, cannot be captured in this fashion. Latter phenomenon means the flux goes from regions of low concentrations to ones with high concentration. Generalization of this approach, analysis and robust numerical algorithms are therefore needed. The fundamental work of Maxwell [17] and Stefan [19] lead to the Maxwell–Stefan diffusion, which uses binary interactions between different species of the mixture. This approach is capable of capturing more complex diffusive effects, but leads to a coupled non-linear set of partial differential equations and is therefore mathematically more challenging. In comparison to Fick’s law the Maxwell–Stefan law is still not studied entirely.

We derive a set of equations which describe the dynamics of an inviscid, isothermal, compressible fluid mixture in a porous medium using a variant of the Maxwell–Stefan theory. The idea to include the porous medium in the model is the same as in the derivation of the dusty gas model [16]. It is regarded as an additional component of the mixture with vanishing velocity and constant density. The solutions of the resulting system automatically satisfy an entropy condition and the second law of thermodynamics. There are some properties known for the system, if only one component is involved. Under certain assumptions there exist smooth solutions to the system globally in time. That means with smooth initial data, the smoothness of the solution does not break down over time. This is a notable characteristic because the system has a hyperbolic structure and formation of discontinuities is common for such systems. Further, existence of a parabolic limit system has been shown in [18]. To be more precise, there exists a parabolic system and the solutions of the original system converge for large times to the solutions of the limit system. In this thesis we investigate whether the aforementioned properties can be transferred to the multicomponent case.

The thesis is structured as follows. In the next section we state in short results for the one component case. Section 3 is dedicated to derive equations which describe the dynamics of an inviscid, isothermal, compressible fluid mixture in a porous medium. We follow the work [3] for this purpose. In section 4 we investigate existence of smooth solutions to the derived system. We introduce a more general setting and cite a theorem from [22] which we then apply to our problem in order to prove the main theorem 4.2. Numerical simulations are performed in section 5. Their purpose is to approve the analytical results we derived in the previous sections. The question of existence of a parabolic limit system is pursued in section 6. Techniques from [14] and [21] are used to prove our main theorem in subsection 6.2. We finally conclude in section 7.
2 One component case

As motivation for the structure of this thesis we briefly state in this section some known facts for the case of one component. We aim to derive similar results for the multicomponent case as well.

Consider inviscid, isothermal compressible fluid flow through porous media. The fluid occupies a region $\Omega \subset \mathbb{R}^d$. The pressure $p = p(\rho) \in \mathbb{R}$ is a function of the mass density.

As mentioned in [13], for one component the system for the mass density $\rho(x,t) > 0$ and velocity $v(x,t) \in \mathbb{R}^d$ reads with $T_f > 0$ as

\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0 & \text{in } \Omega \times (0,T_f) \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v + p(\rho)I) &= -M \rho v & \text{on } \Omega \times \{0\}.
\end{align*}

These are basically damped Euler equations with damping factor $M > 0$. Note that this is the isentropic case. The system (2.1) is a special case of our later results, see system (Euler-Darcy-MS) and the form of the derived entropy production $\zeta$ in theorem 4.2.

With techniques from [12] the following theorem can be proven. It states that for one space dimension and smooth initial data the solution to (2.1) exists globally in time and is smooth.

**Theorem 2.1**

Let $d = 1$ and $p$ be continuously differentiable with

$$p'(\rho) > 0.$$  

Further, let the damping factor $M$ be positive.

The function

$$\bar{U}(x,t) = (\bar{\rho},0)^T, \quad \bar{\rho} > 0.$$  

is an equilibrium solution of the system (2.1).

Then there exists a constant $c_1 \geq 0$, such that if $(\rho_0, v_0^T) =: U_0 \in C^1(\mathbb{R})$ satisfies $\|U_0 - \bar{U}\|_{C^1} \leq c_1$, the system (2.1) with $U_0$ as initial condition has a unique smooth solution $U \in C^1([0, \infty) \times \mathbb{R})$.

Further, in [13] it has been shown, that the solution of (2.1) in one space dimension, i.e.

\begin{align*}
\rho_t + (\rho v)_x &= 0 \\
(\rho v)_t + (\rho v^2 + p(\rho))_x &= -M \rho v,
\end{align*}

(2.2)

converges for large times to the solution of the porous media system

\begin{align*}
\rho_t - (M^{-1} p(\rho)_x)_x &= 0 \\
\rho v + M^{-1} p(\rho)_x &= 0.
\end{align*}

(2.3)

The density obeys the porous medium equation and the momentum Darcy’s law. A special solution for the porous medium equation is the Barenblatt solution. For details on this special solution see [2].
3 Modelling

This section is dedicated to the derivation of a system describing the dynamics of an inviscid, compressible fluid with multiple components. We consider isothermal flow in a porous medium. In the case of porous media and multicomponent gaseous mixtures, the classical Fickian approach for modelling the diffusion phenomena is too simplistic. The flux may not be proportional to the concentration gradient and so called uphill diffusion, where the flux goes from regions of low concentration to ones with high concentration, can not be captured. See [5] and references therein. An example for the breakdown of the Fickian approach is given in [4], where the process of breathing a mixture of helium and oxygen is considered. Duncan and Toor made an experimental example of a three component gas mixture in [8]. Neglecting these facts results in systems, which violate the second law of thermodynamics. In order to derive a system which does not violate this law, we follow the work of Bothe and Dreyer [3] and use a Maxwell–Stefan ansatz to describe the friction between the components.

3.1 Inviscid non-reactive fluid mixtures with Maxwell–Stefan diffusion

Let the fluid with fixed temperature $T_\ast > 0$ consist of $n \in \mathbb{N}$ components $A_1, \ldots, A_n$ with corresponding mass densities $\rho_i = \rho_i(x, t) > 0$ and velocities $\mathbf{v}_i = \mathbf{v}_i(x, t) \in \mathbb{R}^d$, $i = 1, \ldots, n$. The fluid mixture occupies a region $\Omega \subset \mathbb{R}^d$.

The total mass density $\rho$ and barycentric velocity $\mathbf{v}$ are defined as

$$
\rho := \sum_{i=1}^{n} \rho_i \quad \mathbf{v} := \frac{1}{\rho} \sum_{i=1}^{n} \rho_i \mathbf{v}_i.
$$

(3.1)

Further, we define the diffusion velocities

$$
\mathbf{u}_i := \mathbf{v}_i - \mathbf{v} \in \mathbb{R}^d.
$$

Definition 3.1 (Simple mixture)

A mixture is called simple, if the partial pressure

$$
p_i = p_i(\rho_i)
$$

depends only on the corresponding mass density $\rho_i$.

Consider the case of a simple mixture.

We start by formulating the partial balances of total mass and momentum.

The partial balances read for $i = 1, \ldots, n$ as

$$
\partial_t \rho_i + \text{div}(\rho_i \mathbf{v}_i) = 0, \quad (3.2a)
$$

$$
\partial_t (\rho_i \mathbf{v}_i) + \text{div}(\rho_i \mathbf{v}_i \otimes \mathbf{v}_i + p_i(\rho_i) \mathbf{I}) = \mathbf{f}_i. \quad (3.2b)
$$
Here $f_i \in \mathbb{R}^d$ is the momentum production and $I$ is the identity tensor. The vanishing right hand side of (3.2a) results from the fact, that we are not considering mass production, which follows from chemical reactions, i.e. we study non-reactive mixtures. Partial stresses simplify in the case of vanishing viscosity to the partial thermodynamic pressures.

In order to fulfill the conservation law for total momentum, the sum over all momentum productions must vanish. Therefore this law reads as

$$\sum_{i=1}^{n} f_i = 0. \quad (3.3)$$

The crucial part is to include the entropy production $\zeta$ in such a way that the second law of thermodynamics holds true. The term $\zeta$ depends on the solution of the system and describes the production of entropy along the solution trajectory.

For a detailed discussion of the second law of thermodynamics we refer to [3]. The important relations and consequences for our problem are stated below.

With the free energy density $\rho \psi(\rho_1, ..., \rho_n)$ we define chemical potentials $\mu_i$ as

$$\mu_i := \frac{\partial \rho \psi}{\partial \rho_i}. \quad (3.4)$$

For the pressure holds as a consequence of the second law of thermodynamics

$$p = -\rho \psi + \sum_{i=1}^{n} \rho_i \mu_i. \quad (3.5)$$

From (3.4) and (3.5) follows the Gibbs–Duhem equation

$$\rho \psi + p - \sum_{i=1}^{n} \rho_i \mu_i = 0. \quad (3.6)$$

The condition on the entropy production reads as

$$\zeta \geq 0 \quad (3.7)$$

and must hold for all solution trajectories of the system. A solution with $\zeta = 0$ is called thermodynamic equilibrium, whereupon this is a pointwise statement.

For the special case of isothermal fluid mixtures without chemical reactions and with vanishing viscosity, Bothe and Dreyer derived the entropy production, namely

$$\zeta = -\sum_{i=1}^{n} u_i \cdot \left( B_i + \frac{1}{T_s} f_i \right), \quad (3.8)$$

with

$$B_i := \frac{\rho_i}{T_s} \nabla \mu_i - \frac{1}{T_s} \nabla p_i \in \mathbb{R}^d. \quad (3.9)$$
3.1 Inviscid non-reactive fluid mixtures with Maxwell–Stefan diffusion

From (3.6) follows

\[ \sum_{i=1}^{n} B_i = 0. \]  

(3.10)

In order to fulfil the entropy inequality (3.7) one can see with (3.8), that we require

\[ -\sum_{i=1}^{n} u_i \cdot \left( B_i + \frac{1}{T_*} f_i \right) \geq 0. \]  

(3.11)

The sums over \( B_i \) and \( f_i \) vanish, see (3.10),(3.3). We use this fact to eliminate \( B_n \) and \( f_n \) in (3.11) and arrive at

\[ -\sum_{i=1}^{n-1} (u_i - u_n) \cdot \left( B_i + \frac{1}{T_*} f_i \right) \geq 0. \]  

(3.12)

We make the following linear ansatz with the positive definite matrix \((\tau_{ij})_{i,j=1}^{n-1}\), to guarantee that (3.12) holds true:

\[ B_i + \frac{1}{T_*} f_i = -\sum_{j=1}^{n-1} \tau_{ij} (u_j - u_n), \quad i = 1, \ldots, n-1. \]  

(3.13)

With \( \tilde{T} = (\tau_{ij} T)_{i,j=1}^{n-1} \) we can write (3.12) now more compact with (3.13) as

\[ \sum_{\kappa=1}^{n-1} \tilde{u}_\kappa \cdot (\tilde{T} \tilde{u})_\kappa \geq 0. \]  

(3.14)

Here \( \tilde{u} := (u_i - u_n)_{i=1,\ldots,n-1} \in \mathbb{R}^{d(n-1)} \) and \( x_\kappa \in \mathbb{R}^d := (x_{(\kappa-1)d+1}, \ldots, x_{\kappa d}) \). The condition (3.14) is automatically satisfied due to the positive definiteness of \( \tilde{T} \).

To make the term (3.13) symmetric we extend the Maxwell–Stefan matrix \( \tilde{T} \) to a \( n \times n \) matrix with

\[ \tau_{nj} = -\sum_{i=1}^{n-1} \tau_{ij}, \quad j = 1, \ldots, n-1, \quad \tau_{in} = -\sum_{j=1}^{n-1} \tau_{ij}, \quad i = 1, \ldots, n-1. \]  

(3.15)

With the definitions (3.15) we obtain from (3.13)

\[ B_i + \frac{1}{T_*} f_i = -\sum_{j=1}^{n} \tau_{ij} (u_j - u_n), \quad i = 1, \ldots, n, \]  

(3.16)

\[ \sum_{j=1}^{n} \tau_{ij} = 0, \quad i = 1, \ldots, n. \]  

(3.17)
3 Modelling

Thus by replacing \( u_n \) with \( u_i \) in (3.16),

\[
B_i + \frac{1}{T_s} f_i = \sum_{j=1}^{n} \tau_{ij}(u_i - u_j), \quad i = 1, \ldots, n.
\] (3.18)

We consider **binary interactions**. This means \( \tau_{ij} = \tau_{ij}(\rho_i, \rho_j) \to 0 \), for \( \rho_i \rho_j \searrow 0 \).

In this case the matrix \( T = (\tau_{ij})_{i,j=1}^{n} \) is symmetric and positive semi-definite, what is proven in [3].

However, the entropy production (3.8) can now with (3.18) and the symmetry of \( T \) be written as

\[
\zeta = -\sum_{i=1}^{n} u_i \cdot \left( B_i + \frac{1}{T_s} f_i \right) = -\frac{1}{2} \sum_{i,j=1}^{n} \tau_{ij}(u_i - u_j)^2.
\] (3.19)

One can see that \( \tau_{ij} \leq 0 \) for all \( i \neq j \) is necessary to achieve \( \zeta \geq 0 \).

The following ansatz is made to match all the requirements:

\[
\tau_{ij} = -\lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j, \quad \text{with} \quad \lambda_{ij}(\rho_i, \rho_j) = \lambda_{ji}(\rho_j, \rho_i), \quad \lambda_{ij}(\rho_i, \rho_j) \geq 0 \quad (i \neq j). \] (3.20)

The \( \lambda_{ij} \) can be interpreted as friction factors of the components of the fluid mixture.

The constitutive law for thermo-mechanical interactions results from (3.9), (3.18) and (3.20) as

\[
f_i = -\rho_i \nabla \mu_i + \nabla p_i - T_s \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j).
\] (3.21)

Note that in (3.21) the diffusion velocities are replaced by the velocities of the corresponding component.

For simple mixtures holds

\[
\rho \psi = \sum_{i=1}^{n} \rho_i \psi_i(\rho_i), \quad \rho_i \psi_i + p_i = \rho_i \mu_i
\] (3.22)

and it follows with (3.4), (3.22)

\[
f_i = -T_s \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j).
\] (3.23)

With this result the partial momentum balances (3.2b) read as

\[
\partial_t(\rho_i v_i) + \text{div}(\rho_i v_i \otimes v_i + p_i(\rho_i) I) = -T_s \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j).
\] (3.24)
3.2 Including the porous medium in the model

The critical issue in our modelling is that we consider the porous medium as an additional component with velocity \( v_{pm} = 0 \) and density \( \rho_{pm} = \text{const.} \).

There are trivially no equations required in the mass balance and momentum balance, but we need to realize the effects of the porous medium on the other components.

So the sum

\[
-T_s \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j)
\]

rather reads as

\[
-T_s \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j) - T_s \lambda_{i,pm}(\rho_i, \rho_{pm}) \rho_{pm} (v_i - v_{pm}).
\]

We merge the porous medium part with help of \textbf{mobility constants}

\[
M_i = M_i(\rho_i, \rho_{pm}) := T_s \lambda_{i,pm}(\rho_i, \rho_{pm}) \rho_{pm}
\]

into

\[
-M_i \rho_i v_i
\]

to arrive at our final system.

Let \( \Omega \subseteq \mathbb{R}^d \) be open and \( T_f > 0 \). Furthermore let \( U_0 : \Omega \to \mathbb{R}^{n(d+1)} \) and \( g : \partial \Omega \times [0, T_f) \to \mathbb{R}^{n(d+1)} \) be any functions. The function \( g \) needs only to be considered, if \( \Omega \) is bounded.

For brevity we use the notation

\[
U := (\rho_1, ..., \rho_n, \rho_1 v_1, ..., \rho_n v_n).
\]

The Euler–Darcy system with Maxwell–Stefan type diffusion reads as

\[
\begin{align*}
\partial_t \rho_i + \text{div}(\rho_i v_i) &= 0 \\
\partial_t (\rho_i v_i) + \text{div}(\rho_i v_i \otimes v_i + p_i(\rho_i) I) &= -M_i \rho_i v_i - T_s \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j) \\
& \quad \text{in } \Omega \times (0, T_f) \\
U &= U_0 \text{ on } \Omega \times \{t = 0\} \\
U &= g \text{ on } \partial \Omega \times [0, T_f].
\end{align*}
\]

(Euler-Darcy-MS)

We stress that the entropy inequality (3.7) is automatically satisfied along solution trajectories of (Euler-Darcy-MS).
3 Modelling

Remark 3.2.

- Even though \( M_i = M_i(\rho_i, \rho_{pm}) \) depends on the densities \( \rho_i \) and \( \rho_{pm} \), we speak of mobility constants.

- When including the porous medium, the condition (3.17) includes the summand of the porous medium part as well. We neglect this fact. That means precisely that we neglect the effect of the porous medium on the conservation of momentum (3.21) and on the entropy (3.8). For many practical applications this simplification is justified. It allows us to distinguish better between the porous medium part and the coupling between the components \( A_1, ..., A_n \) of the mixture.

- In the case of the absence of coupling, i.e. \( \lambda_{ij} = 0 \) for \( i, j = 1, ..., n \), we have \( n \) decoupled equations of the form (2.1), so the one component system (2.1) results as a special case of our multicomponent system (Euler-Darcy-MS).

- The term "diffusion" might be a bit confusing in this point because one would suspect the appearance of a Laplacian in the equations. The diffusive behaviour of the Maxwell–Stefan terms becomes more clear with the results from section 6, see for instance (6.3).
4 Global classical wellposedness of the Cauchy problem

The main result in this section is the existence of smooth solutions of the system (Euler-Darcy-MS). For this purpose we cite a result of Yong in subsection 4.1, which he derived in [22]. Our problem matches the exact setting, so we apply this theorem in subsection 4.2 to our system and prove main theorem 4.2. For vanishing right hand side hyperbolic balance laws are simply conservation laws. It is known that in this case solutions generally develop singularities in finite time [6]. The theorem of Yong shows under which conditions on the source term, with smooth initial data, the smooth solutions of the system do not break down.

4.1 Yong’s result for general hyperbolic balance laws

Let the state space $G \subset \mathbb{R}^n$ be open and $U : \mathbb{R}^d \times [0, \infty) \to G$ be the unknown function. Further let $Q : G \to \mathbb{R}^n$ and $F_j : G \to \mathbb{R}^n, j = 1, \ldots, d$ be smooth functions. Consider nonlinear systems of balance laws in $d$ space dimensions

$$U_t + \sum_{j=1}^{d} F_j(U) x_j = Q(U). \quad (4.1)$$

In typical applications the source term $Q(U)$ already has, or can be transformed with a linear transformation into the form

$$Q(U) = \begin{pmatrix} 0 \\ q(U) \end{pmatrix},$$

with $q : G \to \mathbb{R}^r$, smooth.

As such applications Yong names for example systems describing non-equilibrium processes for media with hyperbolic response and systems which arise in the numerical solution of conservation laws by relaxation schemes.

From now on we assume that $U$ can be written as $U = (u, w)^T$ and do not distinguish the two forms.

The system (4.1) can be written as

$$\begin{pmatrix} u \\ w \end{pmatrix}_t + \sum_{j=1}^{d} F_j(u, w) x_j = \begin{pmatrix} 0 \\ q(u, w) \end{pmatrix}. \quad (4.2)$$

Note the slight abuse of notation, since $q$ as a function of $U$ is now interpreted as a function of $u$ and $w$. A constant vector $\bar{U} \in G$ is called equilibrium state for the balance law (4.2), if $q(\bar{U}) = 0$. 

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4 Global classical wellposedness of the Cauchy problem

Theorem 4.1 ([22])
Let $s \geq s_0 + 1 = \lfloor d/2 \rfloor + 2$ be an integer and $\bar{U} \in G$ be a constant equilibrium state where the following conditions hold:

i) The Jacobian $q_w(\bar{U}) \in \mathbb{R}^{r \times r}$ is invertible.

ii) There exists a strictly convex smooth function $\eta : G \to \mathbb{R}$, defined in a convex, compact neighbourhood $G$ of $\bar{U}$, such that $\eta_{UU}(U)F_{j,U}(U)$ is symmetric for all $U \in G$ and all $j$. The function $\eta$ is called entropy function.

iii) There is a positive constant $c_G$ such that for all $U \in G$,

$$\left[ \eta(U) - \eta(\bar{U}) \right] Q(U) \leq -c_G |Q(U)|^2,$$

where $|Q|^2 := (Q^T Q)$, denotes the squared euclidean norm of the vector $Q$.

iv) The kernel $\ker(Q_{UU}(\bar{U}))$ of the Jacobian $Q_{UU}(\bar{U})$ contains no eigenvector of the matrix $\sum_j \omega_j F_{j,U}(\bar{U})$ for any $\omega = (\omega_1, \omega_2, ..., \omega_d) \in S^{d-1}$ (the unit sphere in $\mathbb{R}^d$).

Then there are two constants $c_1, c_2$ such that if $U_0 = U_0(x) \in H^s(\mathbb{R}^d)$ satisfies

$$\|U_0 - \bar{U}\|_s \leq c_1,$$

then the system of balance laws (4.2) with $U_0$ as its initial value has a unique global solution $U = U(x, t) \in C([0, \infty); H^s(\mathbb{R}^d))$ satisfying

$$\|U(\cdot, T) - \bar{U}\|^2_s + \int_0^T \|Q(U)(\cdot, t)\|_s^2 dt + \int_0^T \|\nabla U(\cdot, t)\|_{s-1}^2 dt \leq c_2 \|U_0 - \bar{U}\|_s^2$$

for any $T > 0$.

4.2 Application of Yong’s theorem to the problem at hand

Let the matrix $\Lambda = (\lambda_{ij}(\rho_i, \rho_j))_{i,j=1,...,n}$ be symmetric, negative semi-definite

$$\Lambda \preceq 0,$$  \hspace{1cm} (4.3)

and satisfy $\lambda_{ij}(\rho_i, \rho_j) \geq 0$ for $i \neq j$.

For the mass densities $\rho_i = \rho_i(x, t)$ and velocities $v_i = v_i(x, t) \in \mathbb{R}^d, i = 1, ..., n$ recall the system (Euler-Darcy-MS).

We restrict ourselves to the case $d = 1$ and prove global existence of smooth solutions to the Cauchy problem.

Consider equilibrium states of our system of the form

$$\bar{U} = (\bar{\rho}_1, ..., \bar{\rho}_n, 0, ..., 0)^T, \hspace{0.5cm} \bar{\rho}_i > 0, \hspace{0.5cm} i = 1, ..., n.$$  \hspace{1cm} (4.4)
4.2 Application of Yong’s theorem to the problem at hand

**Theorem 4.2** (Global classical wellposedness of the Cauchy problem for (Euler-Darcy-MS))

Let $d = 1$. Further, let $\lambda_{ij} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$, $i, j = 1, ..., n$, $i \neq j$, and $\lambda_{ii} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1, ..., n$. Let $p_i : \mathbb{R} \to \mathbb{R}$, $i = 1, ..., n$ be continuously differentiable with

$$p_i' > 0 \text{ for all } i = 1, ..., n. \quad (4.5)$$

In addition let the mobility constants $M_i > 0$, $i = 1, ..., n$. Consider equilibria $\bar{U}$ of the system (Euler-Darcy-MS) with $n$ constituents $A_1, ..., A_n$, of the form

$$\bar{U} = (\bar{\rho}_1, ..., \bar{\rho}_n, 0, ..., 0)^T, \quad \bar{\rho}_i > 0, \quad i = 1, ..., n.$$

Then there exists a constant $c_1 \geq 0$, such that if $U_0 \in C^1(\mathbb{R})$ satisfies $\|U_0 - \bar{U}\|_{C^1} \leq c_1$, the system (Euler-Darcy-MS) with $U_0$ as initial condition has a unique smooth solution $U \in C^1(\mathbb{R} \times [0, \infty))$.

The entropy inequality

$$\zeta \geq 0,$$

with

$$\zeta := \frac{1}{2} \sum_{i,j=1}^{n} \lambda_{ij}(\rho_i, \rho_j)\rho_i\rho_j(v_i - v_j)^2$$

is satisfied along all solution trajectories. Additionally there exists $c_2 \geq 0$ and it holds the inequality:

$$\|U(\cdot, T) - \bar{U}\|_{C^1}^2 + \int_0^T \|Q(U)(\cdot, t)\|_{C^1}^2 \, dt + \int_0^T \|
abla U(\cdot, t)\|_{C^0}^2 \, dt \leq c_2 \|U_0 - \bar{U}\|_{C^1}^2,$$

for any $T > 0$.

Here $\|f\|_{C^0} := \max_{x \in \mathbb{R}} \|f(x)\|$ and $\|f\|_{C^1} := \|f\|_{C^0} + \|f'\|_{C^0}$.

**Proof.** Throughout the proof we may omit the argument $(\rho_i, \rho_j)$ of the function $\lambda_{ij}$ if this is convenient.

First we cast our problem in the correct setting (4.2) to apply Yong’s theorem 4.1.

For the system (Euler-Darcy-MS) we have $u = (\rho_1, ..., \rho_n)^T$ and $w = (\rho_1 v_1, ..., \rho_n v_n)^T$. Hence, $r = n$.

The function $q$ reads as

$$q(u, w)_i = -M_i \rho_i v_i - T_i \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j)\rho_i\rho_j(v_i - v_j), \quad i = 1, ..., n.$$
i) We obtain with the equilibrium state (4.4)

\[
q_w(\bar{U}) = \begin{pmatrix}
-M_1 - T_1 \sum_{j=1, j \neq 1}^{n} \bar{\rho}_j \lambda_{1j} & T_1 \bar{\rho}_1 \lambda_{12} & \cdots & T_1 \bar{\rho}_1 \lambda_{1n} \\
T_2 \bar{\rho}_2 \lambda_{21} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & T_n \bar{\rho}_n \lambda_{n1} \\
T_n \bar{\rho}_n \lambda_{n1} & \cdots & T_n \bar{\rho}_n \lambda_{n,n-1} & -M_n - T_n \sum_{j=1, j \neq n}^{n} \bar{\rho}_j \lambda_{nj}
\end{pmatrix}
\]

With \( \bar{\rho}_i \lambda_{ii} = -\sum_{j=1, j \neq i}^{n} \bar{\rho}_j \lambda_{ij} \) (3.20), (3.17), this reads as

\[
q_w(\bar{U}) = -M + T_1 R \Lambda,
\]

where \( M = \text{diag}(M_1, \ldots, M_n) \succ 0 \) and \( R = \text{diag}(\bar{\rho}_1, \ldots, \bar{\rho}_n) \succ 0 \).
Since \( T_1 > 0 \) and \( \Lambda \preceq 0 \), the Jacobian \( q_w(\bar{U}) < 0 \) is invertible.

ii) As mathematical entropy function we use a combination of the inner energy (3.22) and the kinetic energy, i.e.

\[
\eta(U) = \sum_{i=1}^{n} \rho_i \psi_i(\rho_i) + \sum_{i=1}^{n} \rho_i \frac{v_i^2}{2}.
\quad (4.7)
\]

For typically strictly convex inner energies \( \psi_i \), this entropy function has the same property, since it is the sum of strictly convex functions.

The choice for \( \eta \) is eligible, if we recall the modelling in section 3.

We have \( \eta_U(U) \in \mathbb{R}^{2n}, \eta_{UU} \in \mathbb{R}^{2n \times 2n} \) and the derivatives read with (3.6) as

\[
\eta_U(U) = \begin{pmatrix}
\psi_1(\rho_1) + \frac{\partial \psi_1}{\partial \rho_1}(\rho_1) \rho_1 - \frac{1}{2} v_1^2 \\
\vdots \\
\psi_n(\rho_n) + \frac{\partial \psi_n}{\partial \rho_n}(\rho_n) \rho_n - \frac{1}{2} v_n^2 \\
0 \\
\vdots \\
v_1 \\
v_n
\end{pmatrix},
\quad \eta_{U}(\bar{U}) = \begin{pmatrix}
\psi_1(\bar{\rho}_1) + \frac{\partial \psi_1}{\partial \bar{\rho}_1}(\bar{\rho}_1) \bar{\rho}_1 \\
\vdots \\
\psi_n(\bar{\rho}_n) + \frac{\partial \psi_n}{\partial \bar{\rho}_n}(\bar{\rho}_n) \bar{\rho}_n \\
0 \\
\vdots \\
v_1 \\
v_n
\end{pmatrix},
\]

\[
\eta_{UU}(U) = \begin{pmatrix}
\text{diag} \left( \frac{1}{\rho_1} p_1'(\rho_1) + \frac{v_1^2}{\rho_1}, \ldots, \frac{1}{\rho_n} p_n'(\rho_n) + \frac{v_n^2}{\rho_n} \right) - \text{diag} \left( \frac{v_1}{\rho_1}, \ldots, \frac{v_n}{\rho_n} \right)
\end{pmatrix},
\]

\[
\quad \text{and} \quad \text{diag} \left( \frac{1}{\rho_1}, \ldots, \frac{1}{\rho_n} \right).
\]
4.2 Application of Yong’s theorem to the problem at hand

With

\[
F(U) = \begin{pmatrix}
\rho_1 v_1 \\
\vdots \\
\rho_n u_n \\
\rho_1 v_1^2 + p_1(\rho_1) \\
\vdots \\
\rho_n v_n^2 + p_n(\rho_n)
\end{pmatrix} \in \mathbb{R}^{2n},
\]

\[
F_U(U) = \begin{pmatrix}
\diagonal(-v_1^2 + p_1'(\rho_1), \ldots, -v_n^2 + p_n'(\rho_n)) & I_n \\
\end{pmatrix} \in \mathbb{R}^{2n \times 2n}
\]

we have

\[
\eta_{UU}(U) F_U(U) = \\
= \begin{pmatrix}
\diagonal(v_1 p_1'(\rho_1) - v_1^3, \ldots, v_n p_n'(\rho_n) - v_n^3) \\
\end{pmatrix} \begin{pmatrix}
p_1'(\rho_1) - v_1^2 \\
p_2'(\rho_2) - v_2^2 \\
\end{pmatrix} \begin{pmatrix}
\diagonal(v_1, \ldots, v_n) \\
\end{pmatrix}
\]

This matrix is obviously symmetric.

iii) We use the symmetry of \( \Lambda \) to rewrite the left hand side of the inequality:

\[
-(\eta_U(U) - \eta_U(\bar{U})) Q(U) = \sum_{i=1}^{n} M_i \rho_i v_i^2 + T_\ast \sum_{i,j=1}^{n} \lambda_{ij} \rho_i \rho_j \frac{(v_i^2 - v_i v_j)}{\rho_i}
\]

\[
= \sum_{i=1}^{n} M_i \rho_i v_i^2 + T_\ast \sum_{i,j=1}^{n} \lambda_{ij} \rho_i \rho_j \left( v_i^2 - v_i v_j \right)
\]

\[
+ \frac{1}{2} T_\ast \sum_{j,i=1}^{n} \rho_i \rho_j \left( v_j^2 - v_j v_i \right)
\]

\[
= \sum_{i=1}^{n} M_i \rho_i v_i^2 + \frac{1}{2} T_\ast \sum_{i,j=1}^{n} \lambda_{ij} \rho_i \rho_j \left( v_i^2 - 2v_i v_j + v_j^2 \right)
\]

\[
= \sum_{i=1}^{n} M_i \rho_i v_i^2 + \frac{1}{2} T_\ast \sum_{i,j=1}^{n} \lambda_{ij} \rho_i \rho_j \left( v_i - v_j \right)^2.
\]
Further,

\[
|Q(U)|^2 = \sum_{i=1}^{n} \left( M_i \rho_i v_i + T_\ast \sum_{j=1}^{n} \lambda_{ij} \rho_i \rho_j (v_i - v_j) \right)^2
\]

\[
\text{with } c = \max_{g,i=1,...,n} M_i \max_{g,i=1,...,n} \rho_i, \quad \ddot{c} = n^2 T_\ast \max_{g,i,j=1,...,n} \lambda_{ij} \left( \max_{g,i=1,...,n} \rho_i \right)^2
\]

\[
|Q(U)|^2 \leq \frac{1}{c_g} \left( \sum_{i=1}^{n} M_i \rho_i v_i^2 + \frac{1}{2} T_\ast \sum_{i,j=1}^{n} \lambda_{ij} \rho_i \rho_j (v_i - v_j)^2 \right), \quad c_g = \frac{1}{2} \min \left\{ \frac{1}{c}, \frac{1}{2\ddot{c}} \right\}.
\]

Hence,

\[
|Q(U)|^2 \leq -\frac{1}{c_g} \left( \eta v(U) - \eta v(\bar{U}) \right) Q(U).
\]

iv)

\[
Q(U) = \begin{pmatrix}
0_n & 0_n \\
-\text{diag} \left( \rho_1 v_1, ..., \rho_n v_n \right) & -T_\ast \mathcal{M} q_w(U)
\end{pmatrix},
\]

with

\[
(\mathcal{M})_{ij} = \begin{cases}
\sum_{j=1, j \neq i}^{n} \left[ \frac{\partial \lambda_{ij} (\rho_i, \rho_j)}{\partial \rho_i} \rho_i \rho_j (v_i - v_j) - \lambda_{ij} (\rho_i, \rho_j) \rho_j v_j \right], & i = j \\
\frac{\partial \lambda_{ij} (\rho_i, \rho_j)}{\partial \rho_j} \rho_i \rho_j (v_i - v_j) + \lambda_{ij} (\rho_i, \rho_j) \rho_i v_i, & i \neq j.
\end{cases}
\]

Therefore with (4.4),

\[
Q(U) = \begin{pmatrix}
0_n & 0_n \\
0_n & q_w(U)
\end{pmatrix}.
\] (4.8)

The lower right block of this matrix is invertible as proven in i), consequently

\[
\ker(Q(U)) = \text{span}\{e_1, ..., e_n\} \subset \mathbb{R}^{2n}.
\]

Here \( e_i = (0, ..., 0, 1, 0, ..., 0)^T \) denotes the \( i \)-th standard unit vector.
4.2 Application of Yong's theorem to the problem at hand

Now we calculate the eigenvectors of $F_U(\bar{U})$

$$F_U(\bar{U}) = \begin{pmatrix} 0_n & I_n \\ \text{diag}(p'_1(\bar{\rho}_1), \ldots, p'_n(\bar{\rho}_n)) & 0_n \end{pmatrix}.$$ 

We partition the vector $x \in \mathbb{R}^{2n}$ accordingly as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$F_U(\bar{U})x = \lambda x \implies x_2 = \lambda x_1 \text{ and } \left( p'_1(\bar{\rho}_1)x_1^{(1)}, \ldots, p'_n(\bar{\rho}_n)x_1^{(n)} \right)^T = \lambda x_2$$

$$\implies \left( p'_1(\bar{\rho}_1)x_1^{(1)}, \ldots, p'_n(\bar{\rho}_n)x_1^{(n)} \right)^T = \lambda^2 x_1 \implies \lambda_{i,n+i} = \pm \sqrt{p'_i(\bar{\rho}_i)}, \ i = 1, \ldots, n.$$

Note that $p'_i(\bar{\rho}_i) > 0$, $i = 1, \ldots, n$. The condition $x_2 = \lambda x_1$ ensures that all eigenvectors of $F_U(\bar{U})$ are not contained in $\ker(Q_U(\bar{U})) = \text{span}\{e_1, \ldots, e_n\}$.

We verified all the conditions of Yong's theorem 4.1 and are now able to apply the same. Hence, the system (Euler-Darcy-MS) with $U_0$ as initial value has a unique global solution $U = U(x,t) \in C([0, \infty), H^s(\mathbb{R}))$, $s \geq 2$.

The Sobolev embedding theorem A.1 gives us $U \in C([0, \infty), C^1(\mathbb{R}))$ and the inequality (3.7) follows directly from theorem 4.1. In particular, from the fact that $U$ is a solution to (Euler-Darcy-MS), it follows that $U \in C^1(\mathbb{R} \times [0, \infty))$.

For the special case $\Lambda = 0$, theorem 4.2 is exactly the corresponding theorem 2.1 for one component.
5 Numerical simulations

In order to get a better understanding of the equations and the derived results so far, we run some numerical simulations in this section. The main goal is to approve the analytical result 4.2 from section 4. Throughout this section we try to add some characteristic time snapshots to illustrate the observations. However, the complete motion pictures are attached to this thesis.

Numerical scheme

All computations were performed with MATLAB®.

We use a finite volume scheme with Roe flux $F$ for our simulations. Let $N$ be the number of components in the fluid mixture. With $U^n_i = (u^n_i, w^n_i)^T \in \mathbb{R}^{2N}$ being the approximated solution at $(x_i, t_n)$, the scheme reads as

$$U^{n+1}_i = (u^n_i, (I_N - M)w^n_i)^T - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2})$$

(5.1)

with $M := \text{diag}(M_1, \ldots, M_N) \in \mathbb{R}^{N \times N}$.

The spatial stepsize $\Delta x$ is fixed and the time stepsize $\Delta t_n := t_{n+1} - t_n$ is computed via the Courant–Friedrichs–Lewy (CFL) condition

$$\lambda_{\text{max}}(J) \frac{\Delta t_n}{\Delta x} \leq \frac{1}{2}$$

(5.2)

in every step $n$. Here $\lambda_{\text{max}}(J)$ denotes the largest absolute value of the eigenvalues of the Jacobian of the flux, depending on the solution at time $t_n$.

If we have the porous medium involved, i.e. $M_i > 0$ for some $i = 1, \ldots, N$, we have to enforce the additional condition

$$\Delta t_n \leq \frac{1}{\max_{x \in [-1,1]} |M_i \rho_i(x, t_n) v_i(x, t_n)|}.$$  

(5.3)

For details on the numerical scheme see [20].

Experiments

Experiments were made for one space dimension ($d = 1$) with one component ($N = 1$) as well as two components ($N = 2$).

The space domain throughout all simulations is

$$\Omega := [-1, 1].$$
5 Numerical simulations

As constitutive equation for the pressure, we use according to the ideal gas law
\[ p_i(\rho_i) = c_i \rho_i, \quad c_i > 0, \quad i = 1, \ldots, N, \quad (5.4) \]
where the constant \( c_i \) remains as additional degree of freedom.

We derive the mathematical entropy. With (3.22) and (3.4) we have
\[ \rho_i \psi_i + p_i = \rho_i \frac{\partial \rho_i \psi_i}{\partial \rho_i}. \quad (5.5) \]
Substituting (5.4) leads to the ordinary differential equation for \( \rho_i \psi_i \)
\[ \rho_i \psi_i + c_i \rho_i = \rho_i \frac{\partial \rho_i \psi_i}{\partial \rho_i}. \quad (5.6) \]
A solution of (5.6) is given by
\[ \rho_i \psi_i = \rho_i (c_i \log(\rho_i) - 1). \quad (5.7) \]
Hence, the mathematical entropy (4.7) reads as
\[ \eta(U) = \sum_{i=1}^{N} \left( \rho_i (c_i \log(\rho_i) - 1) + \rho_i v_i^2 \right). \quad (5.8) \]
We plot the numerical solution \( U \) over time, the integral over \( \Omega \) of the mathematical entropy \( \eta(U) \), to check whether the usual sign condition holds true [10], and \( \sum_{i,j} w_{ij}^{n,(1)} \) over time as well, to check the conservation of momentum. Here the superscript \((j)\) denotes the components of the vector \( w_i^n \).

The conservation of momentum needs to hold true for the case with two components and no porous medium involved, due to the following observation:
Suppose we have smooth solutions of (Euler-Darcy-MS). It follows
\[ (\rho_1 v_1)_t + (\rho_1 v_1^2 + p_1(\rho_1))_x = -T_s \lambda_{12} \rho_1 \rho_2 (v_1 - v_2) \]
\[ (\rho_2 v_2)_t + (\rho_2 v_2^2 + p_2(\rho_2))_x = -T_s \lambda_{21} \rho_2 \rho_1 (v_2 - v_1) \]
Adding both equations yields with \( \lambda_{21} = \lambda_{12} \) (3.20)
\[ (\rho_1 v_1 + \rho_2 v_2)_t + (\rho_1 v_1^2 + \rho_2 v_2^2 + p_1(\rho_1) + p_2(\rho_2))_x = 0 \]
\[ \Rightarrow \frac{d}{dt} \left( \int_{-1}^{1} (\rho_1 v_1 + \rho_2 v_2) \, dx + \left[ t \left( \rho_1 v_1^2 + \rho_2 v_2^2 + p_1(\rho_1) + p_2(\rho_2) \right) \right]_{\rho_{\text{left}}}^{\rho_{\text{right}}} \right) = 0. \]
Since we have implemented periodical boundary conditions, the term \([\ldots]\) drops out and we finally have
\[ \frac{d}{dt} \left( \int_{-1}^{1} \rho_1 v_1 + \rho_2 v_2 \, dx \right) = 0. \]
Translated to the discrete setting that means the sum over all cell values of \( w_i^{n,(1)} + w_i^{n,(2)} \) is constant for every timestep \( t_n \).
5.1 Experiments with one component

At first we consider the case with only one component, \( N = 1 \). The equations (Euler-Darcy-MS) read as

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0 \\
(\rho v)_t + (\rho v^2 + p(\rho))_x &= -M \rho v.
\end{align*}
\] (5.9)

5.1.1 Riemann problem

The constant \( c \) determines primarily the wave speed. We set \( c = 1 \). The time interval where we run our simulation is \([0, 1]\).

The initial condition is established as

\[
U_0 = \begin{cases} 
(1, 0)^T, & x < 0 \\
(0.5, 0)^T, & x \geq 0.
\end{cases}
\]

An equilibrium of this system is

\[
\bar{U} = \begin{pmatrix} 0.75 \\
0 \end{pmatrix}.
\]

We consider a Riemann problem, therefore we choose as boundary conditions Neumann conditions, rather than periodic ones. If we would choose periodic boundary conditions, we would generate a second Riemann problem which overlays the initial one and complicates the observation.
5 Numerical simulations

For the case $M = 0$ we expect shock waves. This can be approved by our simulation, cf. figure 1.

In this figure and all following ones, densities are red depicted, green the momenta and blue the mathematical entropy $\eta$ and the total momentum.

Figure 1: Time snapshots for a Riemann problem of the system (Euler-Darcy-MS) for one component without porous medium ($M = 0$).
If we gradually increase $M$ the solution starts to look smoother. For $M \approx 10^{-2}$ the discontinuity becomes smooth. See figure 2.

Figure 2: Time snapshots for a Riemann problem of the system (Euler-Darcy-MS) for one component and $M = 3E - 2$.

Note that this is more than we have proven with theorem 4.2, because the initial condition is not smooth, but we still have with $M > 0$ a smoothing effect.

We can see that there is no conservation of total momentum, but due to the Neumann boundary condition we did not expect that anyway.

It is possible to increase $M$ up to $M \approx 1$. After that threshold we get problems with our solver. Because of the conditions (5.2), (5.3), our time step size becomes way to small ($< 10^{-14}$) and the solver does not work properly any more.
5 Numerical simulations

5.1.2 Smooth initial data

Now we consider smooth initial data, namely

\[ U_0(x) = \begin{pmatrix} 10 + 9 \sin(3\pi x) \\ 0 \end{pmatrix}. \]

An equilibrium of this system is

\[ \bar{U} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}. \]

Periodic boundary conditions are implemented.
5.1 Experiments with one component

For $M = 0$ we should have conservation of total momentum. In figure 3 we can see some oscillations, but in the range of $10^{-15}$, so up to some numerical errors we have conservation.

Figure 3: Time snapshots of the system (Euler-Darcy-MS) for one component without porous medium ($M = 0$) and smooth initial data.
5 Numerical simulations

Again we gradually increase the mobility constant $M$. The shocks vanish already for $M \approx 3 \cdot 10^{-3}$, but we can still see non-smooth parts in the solution up to $M \approx 5 \cdot 10^{-3}$, where the solution looks smooth. See figure 4, 5.

Figure 4: Time snapshots of the system (Euler-Darcy-MS) for one component, $M = 3E - 3$, and smooth initial data.
5.1 Experiments with one component

Figure 5: Time snapshots of the system (Euler-Darcy-MS) for one component, $M = 5E - 3$, and smooth initial data.
5 Numerical simulations

5.2 Experiments with two components

The previous case with \( N = 1 \) is not that interesting because the modelled Maxwell–Stefan cross diffusion terms do not occur.

Now we consider two components \( (N = 2) \). The corresponding equations (Euler-Darcy-MS) read as

\[
\begin{align*}
(\rho_1)_t + (\rho_1 v_1)_x &= 0 \\
(\rho_2)_t + (\rho_2 v_2)_x &= 0 \\
(\rho_1 v_1)_t + (\rho_1 v_1^2 + p_1(\rho_1))_x &= -M_1 \rho_1 v_1 - T_* \lambda_{12} \rho_1 \rho_2 (v_1 - v_2) \\
(\rho_2 v_2)_t + (\rho_2 v_2^2 + p_2(\rho_2))_x &= -M_2 \rho_2 v_2 - T_* \lambda_{21} \rho_2 \rho_1 (v_2 - v_1).
\end{align*}
\]

5.2.1 Riemann problem

As in the previous subsection 5.1, we start by considering a Riemann problem, namely we enforce the initial condition

\[
U_0 = \begin{cases} 
(1, 0.5, 0, 0)^T, & x < 0 \\
(1.2, 0.6, 0, 0)^T, & x \geq 0.
\end{cases}
\]

An equilibrium of this system is

\[
\bar{U} = \begin{pmatrix} 
1.1 \\
0.55 \\
0 \\
0
\end{pmatrix}.
\]
5.2 Experiments with two components

We would like to see four different shocks in the case
\( M_1 = M_2 = 0 \) and \( 0_{2 \times 2} = \Lambda := \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix} \).

We set \( c_1 = 1.2 \) and \( c_2 = 0.8 \) to get different wave speeds.

One can see the waves with different speeds in figure 6.

Figure 6: Time snapshots for a Riemann problem of the system (Euler-Darcy-MS) for two components without porous medium \( (M_1 = M_2 = 0) \).

If we increase \( M_1, M_2 \) we have the same effect as in the case for \( n = 1 \), but for every component independently, because we have no coupling due to \( \Lambda = 0_{2 \times 2} \).
5 Numerical simulations

5.2.2 Smooth initial data

In this subsection we consider the smooth initial data

\[
U_0(x) = \begin{pmatrix}
1 + 0.9 \sin(3\pi x) \\
0.8 + 0.5 \sin(3\pi x) \\
0 \\
0
\end{pmatrix}.
\]

An equilibrium of this system is

\[
\bar{U} = \begin{pmatrix}
1 \\
0.8 \\
0 \\
0
\end{pmatrix}.
\]

Periodic boundary conditions are implemented.
5.2 Experiments with two components

For $M_1 = M_2 = 0$ and $\lambda = 0$ we make the same observations as in the case for one component because no coupling is present. See figure 7.

Figure 7: Time snapshots of the system (Euler-Darcy-MS) for two components without porous medium ($M_1 = M_2 = 0$), and smooth initial data.

Increasing $M_1/2$ leads to the same effect as in the one component case. We omit the corresponding time snapshots.
5 Numerical simulations

Instead, we increase $\lambda$ but let $M_{1/2} = 0$. Even though we proved the regularity of solutions only for $M \neq 0$, we want to check the necessity of this assumption numerically. It should be noted that we were only able to increase $\lambda$ up to $\lambda \approx 0.5$. For higher values the time step size becomes too small due to the conditions (5.2) and (5.3).

However, for $\lambda \leq 0.5$ we can not observe any regularity of the solution. We depicted the results for $\lambda = 0.5$ in figure 8.

Figure 8: Time snapshots of the system (Euler-Darcy-MS) for two components without porous medium ($M_1 = M_2 = 0$), $\lambda = 0.5$, and smooth initial data.
5.2 Experiments with two components

For the sake of completeness we set in the next experiment $\lambda = 0.2$ and $M_1 = 0.2$, $M_2 = 0.7$. There is no unexpected behaviour of the solution. Even though the mobility constants differ, the solution for each component looks similar. See figure 9. The different constants $c_1$, $c_2$ and the coupling are accountable for this effect.

![Figure 9: Time snapshots of the system (Euler-Darcy-MS) for two components, $M_1 = 0.2$, $M_2 = 0.7$, $\lambda = 0.2$, and smooth initial data.](image)
5 Numerical simulations

As a last experiment we set \( M_1 = 0 \) and \( M_2 = 1 \). Additionally, we include some \( \lambda > 0 \). In this case the matrix \( q_w(\bar{U}) \) from the proof of theorem 4.2 is negative semidefinite and not negative definite.

For \( \lambda \approx 0.4 \) we get oscillations and again problems with the time stepsize. If we include a small \( \lambda = 0.05 \) we observe a smoothing effect, but our solver is not reliable any more, since for example the deviation of the total momentum from zero jumps to \( 10^{-5} \). See figure 10.

Figure 10: Time snapshots of the system (Euler-Darcy-MS) for two components, \( M_1 = 0, M_2 = 1, \lambda = 0.05 \), and smooth initial data.
5.3 Summary

We were able to verify the analytical result from section 4. If the initial data are smooth and close to the equilibrium point of the system, we get for $M_i > 0$ smooth solutions. Moreover, we were able to see a smoothing effect even for non-smooth initial data. The mathematical entropy behaves as expected, i.e. it is non-increasing and the total momentum stays nearly constant in the cases where it should.

If we set in the case of two components one mobility constant to zero and the other to one, with some small $\lambda$ we can observe a smoothing effect as well. But in this case our solver is not trustworthy any more. So this might result from numerical inaccuracy.

For values of $M$ around 1, depending on $\lambda$, the solver was not working properly any more due to the restrictive conditions (5.2), (5.3) on the time step size. We use an explicit solver and suspect that numerical inaccuracy is accountable for non-smoothness in case of small mobility constants.
6 Parabolic asymptotic limit of the Euler–Darcy–MS system

The goal of this section is to prove the convergence of the solutions to (Euler-Darcy-MS) in $C^1$ to the solution of a parabolic limit system. We follow strictly [14] and [21] in order to achieve this goal, with some modifications, because our system does not match the exact setting of these works.

Throughout this section let

$$M_1, \ldots, M_n > 0. \quad (6.1)$$

In order to perform asymptotic analysis on our system (Euler-Darcy-MS), we make a non-dimensionalization of the same.

For this purpose let $\bar{x}, \bar{t}, \bar{\rho} > 0, \bar{v}, \bar{p}, \bar{M},$ and $\bar{\lambda}$ be the characteristic scales of the corresponding quantities.

We introduce the rescaled variables $\hat{x}_1, \ldots, \hat{x}_d, \hat{t}$ through

$$\bar{x} \cdot \hat{x} = x, \bar{t} \cdot \hat{t} = t.$$ 

The other dimensionless variables $\hat{\rho}, \hat{p}, \hat{M},$ and $\hat{\lambda}$ are defined in the same fashion.

Observe that

$$\hat{\nabla} = \bar{x} \nabla, \quad \hat{\Delta} = \bar{x}^2 \Delta, \quad \hat{\text{div}} = \bar{x} \text{div} \quad \text{and} \quad \frac{\partial}{\partial \hat{t}} = \bar{t} \frac{\partial}{\partial t}.$$ 

We suppress the $\hat{\ }$ notation and rewrite (Euler-Darcy-MS) into the form

$$A \partial_t \rho_i + \text{div}(\rho_i v_i) = 0$$

$$A \partial_t (\rho_i v_i) + \text{div}(\rho_i v_i \otimes v_i) + (\bar{Ma})^{-2} \text{div}(p_i(\rho_i)I) = \bar{M} \bar{x} \frac{\bar{\rho} \bar{v}}{\bar{v}} M_i \rho_i v_i - T_s \frac{\bar{\lambda} \bar{\rho} \bar{x}}{\bar{v}} \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j). \quad \text{(Euler-Darcy-MS-ND)}$$

Here $A := \bar{x} / \bar{v} t$ is a characteristic number and $\bar{Ma} := \sqrt{\bar{v}^2 \bar{p}} / \bar{p}$ is the Mach number.

We assume that the characteristic scales have the following asymptotic behaviour.

**Assumption 6.1.** Let

$$A = O(\varepsilon), \quad (\bar{Ma})^{-2} = O(1), \quad \frac{\bar{M} \bar{x}}{\bar{v}} = O(\varepsilon^{-1}) \quad \text{and} \quad \frac{\bar{\lambda} \bar{\rho} \bar{x}}{\bar{v}} = O(\varepsilon^{-1}). \quad \text{(AS)}$$

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6 Parabolic asymptotic limit of the Euler–Darcy–MS system

In this regime the system (Euler-Darcy-MS-ND) reads as

\[
\begin{align*}
\varepsilon \partial_t \rho_i + \text{div}(\rho_i v_i) &= 0 \\
\varepsilon \partial_t (\rho_i v_i) + \text{div}(\rho_i \otimes v_i) + \text{div}(p_i(\rho_i) I) &= 0 \\
&= -\frac{1}{\varepsilon} M_i \rho_i v_i - \frac{T^*}{\varepsilon} \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \rho_j (v_i - v_j).
\end{align*}
\]

(6.2)

We conjecture that the solutions of the system (Euler-Darcy-MS-ND) converge for \(\varepsilon \to 0\) to the solutions of

\[
\begin{align*}
\mathbf{r}_t + (\mathbf{B}^{-1} p_x)_x &= 0
\end{align*}
\]

(6.3)

with

\[
\begin{align*}
\mathbf{r} &= (\rho_1, ..., \rho_n)^T, \mathbf{p} = (p_1(\rho_1), ..., p_n(\rho_n))^T
\end{align*}
\]

and

\[
\mathbf{B} = \text{diag}(-M_1, ..., -M_n) + T_s \text{diag}(\rho_1, ..., \rho_n) \Lambda.
\]

The next subsections are dedicated to the task of proving this conjecture rigorously. However, we first note the following property of the system (6.3).

**Lemma 6.2**
The system (6.3) is parabolic.

**Proof.** We use chainrule to rewrite the system (6.3) into

\[
\begin{align*}
\mathbf{r}_t + (\tilde{\mathbf{B}}^{-1} \mathbf{r}_x)_x &= 0,
\end{align*}
\]

(6.4)

with \(\tilde{\mathbf{B}} = \text{diag}(p'_1(\rho_1), ..., p'_n(\rho_n))^{-1} \cdot \mathbf{B}\).

In order to prove the parabolicity of the system, we need to show that \(\tilde{\mathbf{B}}^{-1}\) exists and is negative definite. This follows from

\[
\text{eig}(\tilde{\mathbf{B}}^{-1}) \subset (-\infty, 0) \iff \text{eig}(\tilde{\mathbf{B}}) \subset (-\infty, 0)
\]

and with (4.5), (4.3), (6.1)

\[
\tilde{\mathbf{B}} = \left[ \begin{array}{c} \text{diag}(p'_1(\rho_1), ..., p'_n(\rho_n))^{-1} \text{diag}(-M_1, ..., -M_n) \\ \frac{T^*}{\varepsilon} \text{diag}(\rho_1, ..., \rho_n) \Lambda \end{array} \right] \prec 0.
\]

\(\square\)
6.1 General framework

Remark 6.3. There are other regimes where the same arguments can be applied. For instance let

\[ A = O(1), \quad (Ma)^{-2} = O(\varepsilon^{-2}), \quad \frac{\bar{M} \bar{x}}{\bar{v}} = O(\varepsilon^{-2}), \quad \text{and} \quad \frac{\bar{\lambda} \bar{p} \bar{x}}{\bar{v}} = O(\varepsilon^{-2}). \]

This results in

\[ \partial_t \rho_i + \text{div}(m_i) = 0 \]

\[ \varepsilon \partial_t (m_i) + \varepsilon \text{div} \left( \frac{m_i \otimes m_i}{\rho_i} \right) + \frac{1}{\varepsilon} \text{div}(p_i(\rho_i) I) = 0 \]

\[ = -\frac{1}{\varepsilon} M_i m_i - \frac{T_s}{\varepsilon} \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) m_i + \frac{T_s}{\varepsilon} \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i m_j. \]

Here \( m_i := \rho_i v_i \). Substitute \( \tilde{m}_i = \varepsilon m_i \) to obtain

\[ \partial_t \rho_i + \frac{1}{\varepsilon} \text{div}(\tilde{m}_i) = 0 \]

\[ \partial_t (\tilde{m}_i) + \frac{1}{\varepsilon} \text{div} \left( \frac{\tilde{m}_i \otimes \tilde{m}_i}{\rho_i} \right) + \frac{1}{\varepsilon} \text{div}(p_i(\rho_i) I) = 0 \]

\[ = -\frac{1}{\varepsilon^2} M_i \tilde{m}_i - \frac{T_s}{\varepsilon^2} \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i m_j + \frac{T_s}{\varepsilon^2} \sum_{j=1}^{n} \lambda_{ij}(\rho_i, \rho_j) \rho_i \tilde{m}_j. \]

This system has the same structure as (6.2). The substitution is feasible because the limit system only involves the densities, not the momenta.

6.1 General framework

In order to prove the convergence of the solutions, we first regard a more general framework. Consider systems in one space dimension with a small parameter \( \varepsilon > 0 \) of the form:

\[ U_t + \frac{1}{\varepsilon} A(U) U_x = \frac{1}{\varepsilon^2} Q(U) \]

\[ U(x, 0) = U_0(x; \varepsilon). \]

(6.5)

The unknown \( U \) is a \( n \)-vector function of \( (x, t) \in \mathbb{R} \times [0, \infty) \), \( A(U) \) and \( B(U) \) are smooth \( n \times n \)-matrix functions and \( Q(U) \) a smooth \( n \)-vector function of \( U \in G \), with \( G \) being a open set in \( \mathbb{R}^n \). Note the similarities to subsection 4.1.

We are interested in the behaviour for \( \varepsilon \to 0 \).

The goal is to construct formal asymptotic approximations

\[ U^m_\varepsilon = \sum_{k=0}^{m} \varepsilon^k O_k(x, t), \]

(6.6)

with \( m \in \mathbb{N} \).
We prove that the solution $U^\varepsilon(x,t)$ to (6.5) exists and can be written as $U^\varepsilon = U^m_\varepsilon + O(\varepsilon^{m+1})$ in the Sobolev space $H^s$. The solution $U^\varepsilon(x,t)$ converges to the leading term $O_0(x,t)$ of $U^m_\varepsilon$ for $\varepsilon \to 0$.

Again, similar to subsection 4.1, we assume that we have the following form of the occurring terms:

$$Q(U) = \begin{pmatrix} 0 \\ q(U) \end{pmatrix}, \quad U = \begin{pmatrix} u \\ w \end{pmatrix},$$

(6.7)

with $q : G \to \mathbb{R}^r$ smooth and the special form of the matrices:

$$A = \begin{pmatrix} 0 & A^{12} \\ A^{21}(U) & A^{22}(U) \end{pmatrix},$$

(6.8)

In addition to that we make the following parabolic structural assumption as in [14]:

$q(u, w) = 0 \iff w = 0$ and $q_w(u, 0)$ is invertible for any $u$ under consideration. (PS)

6.1.1 Formal asymptotic approximations

We want to construct a solution, called outer expansion, of the form

$$O_\varepsilon(x,t) = \sum_{k=0}^{\infty} \varepsilon^k O_k(x,t).$$

(6.9)

Note that such a solution cannot in general satisfy the initial data. We assume the initial data have a form such that (6.9) can fulfil them, i.e. $Q(U_0(x;0)) = 0$ (see (6.14a) below).

Without this assumption we would need another expansion, the inner expansion

$$I_\varepsilon(x,\tau) = \sum_{k=0}^{\infty} \varepsilon^k I_k(x,\tau),$$

(6.10)

with inner variable $\tau = t/\varepsilon^2$. The purpose of this expansion is taking the initial value. After deriving the inner expansion, the two expansions need to be matched with the matching principle of [9]. The proof becomes more technical in this case.

In the following we derive equations for the outer expansion (6.9). We use formal asymptotic expansions

$$A \left( \sum_{k=0}^{\infty} \varepsilon^k O_k \right) = A(O_0) + \sum_{k=1}^{\infty} \varepsilon^k [A_U(O_0) \cdot O_k + C(A, k, O)],$$

(6.11)

where $C(A, k, O)$ is determined by $A$ and the first $k$ components of the sequence $O := (O_0, O_1, O_2, \ldots)$. Further, $C(A, 1, O) = 0$. 

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6.1 General framework

Introduce the residual

\[ R(U) := U_t + \frac{1}{\varepsilon} A(U) U_x - \frac{1}{\varepsilon^2} Q(U). \] (6.12)

We need (6.9) to be a formal solution of (6.5), hence

\[ R \left( \sum_{k=0}^{\infty} \varepsilon^k O_k \right) \frac{1}{\varepsilon} = 0. \]

We infer

\[ R \left( \sum_{k=0}^{\infty} \varepsilon^k O_k \right) = \sum_{k=0}^{\infty} \varepsilon^k \left[ A(O_0) + \sum_{k=1}^{\infty} \varepsilon^k [A_U(O_0) \cdot O_k + C(A, k, Q)] \right] \sum_{k=0}^{\infty} \varepsilon^k O_{k,x} \]

\[ - \frac{1}{\varepsilon^2} \left[ Q(O_0) + \sum_{k=1}^{\infty} \varepsilon^k [Q_U(O_0) O_k + C(Q, k, Q)] \right]. \]

Thus,

\[ R \left( \sum_{k=0}^{\infty} \varepsilon^k O_k \right) = -\varepsilon^{-2} Q(O_0) + \varepsilon^{-1} [A(O_0) O_{0,x} - Q_U(O_0) O_1] \]

\[ + \sum_{k=0}^{\infty} \varepsilon^k [O_{k,t} + A(O_0) O_{k+1,x} - Q_U(O_0) O_{k+2}] \]

\[ + \sum_{i=1}^{k+1} (A_U(O_0) \cdot O_i + C(A, i, Q)) O_{k+1-i,x} - C(Q, k + 2, Q) \]

\[ \frac{1}{\varepsilon} = 0. \] (6.13)

The \( u \)-part of \( C(Q, k, Q) \) drops out for all \( k \) because it vanishes for \( Q \) itself. We infer from (6.13)

\[ Q(O_0) = 0, \] (6.14a)

\[ A(O_0) O_{0,x} = Q_U(O_0) O_1, \] (6.14b)

\[ O_{k,t} + A(O_0) O_{k+1,x} - Q_U(O_0) O_{k+2} \]

\[ + \sum_{i=1}^{k+1} (A_U(O_0) \cdot O_i + C(A, i, Q)) O_{k+1-i,x} - C(Q, k + 2, Q) = 0, \ k \geq 0. \] (6.14c)

We set \( O_k = (u^T_k, w^T_k)^T \). With (6.7) the equation (6.14a) reads as

\[ q(u_0, w_0) = 0. \] (6.15)

From (6.15) follows that the leading term \( O_0 \) of the outer expansion \( W^m \) is in equilibrium. With (PS) we obtain

\[ w_0 = 0, \ q_u(u_0, w_0) = 0. \] (6.16)
Since $Q_U$ has a special structure, namely, $Q_U(u, w) = \begin{pmatrix} 0 & 0 \\ q_w(u, w) & q_u(u, w) \end{pmatrix}$, equation (6.14b) reads as

$$A(O_0)^{21}u_{0,x} = q_w(u_0, 0)w_1.$$  

(6.17)

This leads to

$$w_1 = q_w(u_0, 0)^{-1}A(O_0)^{21}u_{0,x}.$$  

(6.18)

For $k = 0$ equation (6.14c) becomes

$$O_{0,t} + A(O_0)O_{1,x} - Q_U(O_0)O_2 + A_U(O_0) \cdot O_1O_{0,x} - C(Q, 2, Q) = 0.$$  

(6.19)

With $w_0 = 0$ the $u$-part of (6.19) yields

$$u_{0,t} + A^{12}(O_0)w_{1,x} = 0.$$  

(6.20)

Hence with (6.20) and (6.18)

$$u_{0,t} + A^{12}(O_0)(q_w(u_0, 0)^{-1}A(O_0)^{21}u_{0,x})_x = 0.$$  

(6.21)

Note that the matrix $A_U(O_0) \cdot O_1$ has the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$, therefore $A_U(O_0) \cdot O_1O_{0,x}$ has no $u$ component.

For our Maxwell–Stefan system (6.2) we have

$$A(r, m) = \begin{pmatrix} \text{diag} \left( \frac{1}{\rho_1 v_1} \frac{1}{\rho_2 v_2} \right) & \frac{I}{\rho_1} \frac{I}{\rho_2} \\ \frac{1}{\rho_2 v_2} \frac{1}{\rho_2 v_2} & \frac{I}{\rho_1} \frac{I}{\rho_2} \end{pmatrix}$$  

(6.22)

$$q(r, m) = \begin{pmatrix} -M_i \rho_i v_i - T \sum_{j=1}^{n} \lambda_{ij} \rho_i \rho_j (v_i - v_j) \\ \end{pmatrix}_{i=1, \ldots, n}.$$  

(6.23)

With (6.22) and (6.23) the system (6.21) reads as

$$r_{0,t} + [B(r_0, 0)^{-1} \text{diag}(p_i'(\rho_0))](r_{0,x})_x = 0,$$  

(6.24)

with $r_0 = (\rho_1, \ldots, \rho_n)^T$. This is for the first order exactly (6.3) as we conjectured. If we are able to show that the solution of (6.2) converges for $\varepsilon \to 0$ to the first order term $O_0$, we proved the assertion.

**Remark 6.4.** We only derived equations for $u_0, w_0$, and $w_1$. To construct the outer expansion entirely, we need equations for all terms. With help of (PS) equations for all other terms can be derived. Assume to have equations for $u_l, w_l, w_{l+1}$ with $l = 0, \ldots, k$. Then one needs to consider the $w$-part of (6.14c). This leads to an expression for $w_{k+2}$ depending only on known terms. Plugging this relation into the $u$-part of (6.14c) yields an equation for $u_{k+1}$. Of course the initial data need to be satisfied, so we have the additional condition on the coefficients $O_k$

$$O_k(\cdot, 0) = U_{0,k}(\cdot), \quad k \geq 0.$$
6.1 General framework

### 6.1.2 Justification of formal approximations

At first we cite some calculus inequalities in Sobolev spaces \([15]\) and a nonlinear Gronwall-type inequality. The proof thereof can be found in \([21]\) and references therein.

**Lemma 6.5**

Let \(s, s_1, s_2 \in \mathbb{N}_0\) and \(s_0 = \lfloor d/2 \rfloor + 1\).

i) If \(s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \geq 0\), then \(H^{s_1}H^{s_2} \subset H^{s_3}\), where the embedding is continuous.

ii) Suppose \(s \geq s_0 + 1\), \(A \in H^s\) and \(U \in H^{s-1}\). Then for all multi-indices \(\alpha\) with \(|\alpha| \leq s\), \(\partial^\alpha(AU) - A\partial^\alpha U \in L^2\) and

\[
\|\partial^\alpha(AU) - A\partial^\alpha U\| \leq C_s\|A\|_s\|U\|_{\|\alpha\|-1}.
\]

iii) Suppose \(s \geq s_0\), \(A \in C^s_b(G)\) and \(V \in H^s(\Omega, G)\). Then \(A(V(\cdot)) \in H^s\) and

\[
\|A(V(\cdot))\|_s \leq C_s|A|_s(1 + \|V\|_s^s).
\]

Here \(C_s\) is a generic constant depending on \(s\) and \(d\) and \(|A|_s := \sup_{U \in G, |\alpha| \leq s} |\partial^\alpha A(U)|\).

**Lemma 6.6** \((\text{[21]})\)

Suppose \(\psi(t)\) is a positive \(C^1\)-function of \(t \in [0, T)\) with \(T \leq \infty\), \(m > 1\) and \(b_1(t), b_2(t)\) are integrable on \([0, T)\). If

\[
\psi'(t) \leq b_2(t)\psi^m(t) + b_1(t)\psi(t),
\]

then there exists \(\delta > 0\), depending only on \(m, C_{1b}\) and \(C_{2b}\), such that

\[
\sup_{t \in [0, T)} \psi(t) \leq e^{C_{1b}},
\]

whenever \(\psi(0) \in (0, \delta]\). Here

\[
C_{1b} = \sup_{t \in [0, T)} \int_t^T b_1(s) \, ds \quad \text{and} \quad C_{2b} = \int_0^T \max\{b_2(t), 0\} \, dt.
\]
We require the following **stability condition**

i) There exists a symmetric positive definite matrix $A_0(U) > 0$ such that

$$A_0(U)A(U) = A^T(U)A_0(U)$$

for all $U \in G$.

ii) There exists a symmetric positive $r \times r$ matrix $S(U)$ such that

$$A_0(U)Q_U(U) + Q^T_U(U)A_0(U) \leq - \begin{pmatrix} 0 & 0 \\ 0 & S(U) \end{pmatrix},$$

for any $U$ with $Q(U) = 0$.

The matrices $A_0(U)$ and $S(U)$ are assumed to be smooth.

Before we prove the validity of the constructed approximations, we make the following assumption with $s > 1 \in \mathbb{N}$. The parts 1)-5) are the same as in [14], the assumption 6) results from our slightly different setting.

**Assumption 6.7.**

1) $A, Q \in C^\infty(G)$, there exists a convex, open set $G_0$ with $G_0 \subset \subset G$ such that $U_0(\cdot; \varepsilon) \in G_0$ for all $\varepsilon > 0$ and all $x \in \Omega = (0, 1]$, and $U_0(\cdot; \varepsilon) \in H^s$ is periodic in $x$ with period 1.

2) $\|U^m_\varepsilon(\cdot, 0) - U_0(\cdot; \varepsilon)\|_s = O(\varepsilon^m)$

3) $O_0$ takes values in $G_0$ and belongs to $C([0, T_m], H^s)$ with $T_m > 0$ a constant depending on $m$.

4) $Q_U(O_0)O_{m+1} \in C([0, T_m], H^s)$

5) $U^m_\varepsilon$ takes values in $G_0$ and $\|U^m_\varepsilon(\cdot, t)\|_{s+1}$ is uniformly bounded with respect to $\varepsilon > 0$ and $t \in [0, T_m]$.

6) The matrices $A(U)$ and $A_0(U)$ have the following special structure:

$$A_0(U^\varepsilon) = \begin{pmatrix} A_{11}^{0} & \varepsilon^2 A_{12}^{0}(\tilde{U}^\varepsilon) \\ \varepsilon^2 A_{12}^{0}(\tilde{U}^\varepsilon) & A_{22}^{0} \end{pmatrix}$$

$$A(U^\varepsilon) = \begin{pmatrix} A_{12} & \varepsilon^2 A_{22}(\tilde{U}^\varepsilon) \\ \varepsilon^2 A_{12}(\tilde{U}^\varepsilon) & A_{22} \end{pmatrix}$$

$$A(U^m_\varepsilon) = \begin{pmatrix} A_{12} & \varepsilon^2 A_{22}(\tilde{U}^m_\varepsilon) \\ \varepsilon^2 A_{12}(\tilde{U}^m_\varepsilon) & A_{22} \end{pmatrix}. $$

Here $\tilde{w}^\varepsilon = w^\varepsilon/\varepsilon^2$, $\tilde{w}^m_\varepsilon = w^m_\varepsilon/\varepsilon^2$, $\tilde{w}^\varepsilon = O(1)$, $\tilde{w}^m_\varepsilon = O(1)$, and $\tilde{U}^\varepsilon = ((u^\varepsilon)^T, (\tilde{w}^\varepsilon)^T)^T$, $\tilde{U}^m_\varepsilon = ((u^m_\varepsilon)^T, (\tilde{w}^m_\varepsilon)^T)^T$. 


Remark 6.8.

i) We demand periodicity of the initial data but the same results hold also for initial data with compact support.

ii) Recall that \( O_0 = (u_0^T, 0)^T \) and \( u_0 \) satisfies (6.21). It can be shown that (6.21) is a parabolic system ([14]). The assumption 3) is fulfilled by using (local-in-time) existence arguments for parabolic systems. The same is true for 4) and 5) since \( O_k \) are also solutions to parabolic systems.

iii) Note that we have trivially

\[
\left\| U^m(\cdot, t) - O_0(\cdot, t) \right\|_s = \left\| \sum_{k=1}^m \varepsilon^k O_k(\cdot, t) \right\|_s = O(\varepsilon). \tag{6.26}
\]

We formulate a lemma for the residual of the formal approximation \( U^m_\varepsilon \).

**Lemma 6.9**

For the formal asymptotic approximation \( U^m_\varepsilon \) from (6.6) it holds

\[
R(U^m_\varepsilon) = \varepsilon^{m-1} Q_U(O_0) O_{m+1} + \varepsilon^{m-1} F_m. \tag{6.27}
\]

Here \( Q_U(O_0) O_{m+1} \) is determined by the first \( m \) terms \( U^m_\varepsilon \) and \( F_m \) satisfies

\[
\left\| F_m \right\|_s \leq C\varepsilon. \tag{6.28}
\]

**Proof.** From (6.13) it follows with (6.14)

\[
R(U^m_\varepsilon) = R\left( \sum_{k=0}^m \varepsilon^k O_k \right) = \varepsilon^{m-1} Q_U(O_0) O_{m+1} + O(\varepsilon^m), \tag{6.29}
\]

where

\[
Q_U(O_0) O_{m+1} = O_{m-1,t} + A(O_0) O_{m,x} + C'_{m+1}(O).
\]

The term \( C'_{m+1} \) is determined by the first \( m + 2 \) elements of the sequence \( O \) and by the first-order derivatives of the first \( m + 1 \) elements.

Introduce \( F_m \) with

\[
\varepsilon^{m-1} F_m = R(U^m_\varepsilon) - \varepsilon^{m-1} Q_U(O_0) O_{m+1}.
\]

The inequality (6.28) is trivial. \( \square \)

**Remark 6.10.** For fixed \( \varepsilon > 0 \) under our assumptions there exists \( T_\varepsilon > 0 \) for any convex open set \( G_1 \) with \( G_0 \subset \subset G_1 \subset \subset G \), such that the unique solution \( U^\varepsilon \) exists. Further, it holds \( U^\varepsilon(x, t) \in \tilde{G}_1 \), for \( (x, t) \in \Omega \times [0, T_\varepsilon] \) and \( U^\varepsilon \in C([0, T_\varepsilon], H^s) \). Let \( [0, T_\varepsilon] \) be the maximal where the solution exists. In [21] it is shown that under the assumptions of the following theorem 6.11 it holds \( T_\varepsilon \geq T_m \). We do not perform the corresponding proof in detail, because in our later application this is not an issue since we already proved global-in-time existence of smooth solutions to our system (6.2) in section 4.

We now come to the main theorem in this section (cf. Theorem 4.2 in [14]).
Theorem 6.11
Let the parabolic structure assumption (PS), the stability condition, and the assumption 6.7 with \( m > 2 \) hold. Further, let \( \mathbb{N} \ni s > 1 \) and \([0, T_{\varepsilon}]\) be the maximal time interval where (6.5) has a solution \( U_{\varepsilon} \in C([0, T_{\varepsilon}], H^s) \) with values in a convex set \( \bar{G}_1 \subset \subset G \).
Suppose \([0, T_m]\) is a time interval where the asymptotic approximation \( U_m^{\varepsilon} \) from (6.9) is well-defined. Then there exists a constant \( K > 0 \) dependent on \( T_m \), such that
\[
\| U_m^{\varepsilon}(t) - U_{\varepsilon}(t) \|_s \leq K \varepsilon^m
\]
for sufficiently small \( \varepsilon \) and \( t \in [0, \min\{T_m, T_{\varepsilon}\}] \).

Proof. The equations (6.5) and (6.27) yield for the error \( E := U_m^{\varepsilon} - U_{\varepsilon} \)
\[
E_t + \frac{1}{\varepsilon} A(U_{\varepsilon}) E_x = \frac{Q(U_m^{\varepsilon}) - Q(U_{\varepsilon})}{\varepsilon^2} + \varepsilon^{m-1} Q_U(O_0) O_{m+1} + \varepsilon^{m-1} F_m
\]
\[
+ \frac{1}{\varepsilon} (A(U_{\varepsilon}) - A(U_m^{\varepsilon})) U_m^{\varepsilon,x}.
\]
We infer by differentiation of this equation with \( \partial^\alpha \) in \( x \), for \( \mathbb{N} \ni \alpha < s \)
\[
E_{\alpha,t} + \frac{1}{\varepsilon} A(U_{\varepsilon}) E_{\alpha,x} = \frac{Q_U(O_0) E_{\alpha}}{\varepsilon^2} + F_1^\alpha + F_2^\alpha,
\]
where
\[
F_1^\alpha = \varepsilon^{m-1} (Q_U(O_0) O_{m+1})_\alpha + \frac{(Q_U(O_0) E_{\alpha} - Q_U(O_0) E_{\alpha})}{\varepsilon^2}
\]
\[
+ \frac{(Q(U_m^{\varepsilon}) - Q(U_{\varepsilon}) - Q_U(O_0) E_{\alpha})}{\varepsilon^2},
\]
\[
F_2^\alpha = \frac{1}{\varepsilon} \left( [A(U_{\varepsilon}) - A(U_m^{\varepsilon})] U_m^{\varepsilon,x} \right)_\alpha
\]
\[
+ \frac{1}{\varepsilon} (A(U_{\varepsilon}) E_{\alpha,x} - (A(U_{\varepsilon}) E_{\alpha})_\alpha)
\]
\[
+ \varepsilon^{m-1} (F_m)_\alpha
\]
\[
=: f_1^\alpha + f_2^\alpha + f_3^\alpha.
\]
For improved structure of the proof, we divide the same in several lemmata.

Lemma 6.12
Under the conditions of theorem 6.11 it holds
\[
\frac{d}{dt} \int_{\Omega} e(E_{\alpha}) \, dx + \frac{c_0}{\varepsilon^2} \| E_{\alpha}^{H} \|_2^2 \leq C \| E_{\alpha}^{H} \| \| F_1^\alpha \| + C \varepsilon \| E_{\alpha} \| \| F_1^\alpha \|
\]
\[
+ C \| E_{\alpha} \|^2 + C \| E_{\alpha} \| \| F_2^\alpha \|
\]
where \( e(E_{\alpha}) = E_{\alpha}^T A_0(U_{\varepsilon}) E_{\alpha} \), \( c_0 = \inf_{U \in \bar{G}_1} |S(U)| \), with \( S \) from the stability condition, and \( C \) is a generic constant.
Proof. The matrix \( A_0(U^\varepsilon)A(U^\varepsilon) \) is symmetric. Hence, for example
\[
(E^T_\alpha A_0(U^\varepsilon)A(U^\varepsilon)E_\alpha)_t = 2E^T_\alpha A_0(U^\varepsilon)A(U^\varepsilon)E_{\alpha,t} + E^T_\alpha (\partial_t A_0(U^\varepsilon)A(U^\varepsilon))E_\alpha.
\]
We multiply equation (6.30) from left with \( E^T_\alpha A_0(U^\varepsilon) \) and obtain with relations like the one given above
\[
e(\alpha_t + \frac{1}{\varepsilon}(E^T_\alpha A_0(U^\varepsilon)A(U^\varepsilon)E_\alpha)x
\]
\[
= \frac{2}{\varepsilon^2}E^T_\alpha A_0(\varepsilon)Q_{\varepsilon}(\varepsilon)E_\alpha + 2E^T_\alpha A_0(\varepsilon)F^\alpha_1 + 2E^T_\alpha A_0(\varepsilon)F^\alpha_2
\]
\[
+ E^T_\alpha (\partial_t A_0(\varepsilon) + \left[ \frac{1}{\varepsilon} A_0(\varepsilon)A(U^\varepsilon) \right]_x) E_\alpha
\]
\[
+ 2\varepsilon E^T_\alpha (A(\varepsilon) - A_0(\varepsilon)) Q_{\varepsilon}(\varepsilon) E_\alpha
\]
\[
+ 2E^T_\alpha (A(\varepsilon) - A_0(\varepsilon)) F^\alpha_1
\]
\[
= : I^\alpha_1 + I^\alpha_2 + I^\alpha_3 + I^\alpha_4 + I^\alpha_5 + I^\alpha_6.
\]
\[
(6.31)
\]
We want to find estimates for each term of the right hand side.

It holds \( Q(\varepsilon) = 0 \) (6.16). Therefore the second part of the stability condition can be applied for \( I^\alpha_1 \) and it follows
\[
I^\alpha_1 \leq \frac{c_0}{\varepsilon^2} |E^T_{\alpha_1}|^2.
\]
The superscript \( I \) denotes the first \( n-r \) components of a vector and \( II \) the last \( r \) components. The \( u \)-component of \( F^\alpha_1 \) vanishes because it does for \( Q(U) \). It can be shown ([14]) that \( A_0(\varepsilon) \) needs to be block-diagonal and indeed in our later application of the theorem this is true, so in \( I^\alpha_2 \) the \( E^T_{\alpha_1} \)-term drops out and we have
\[
I^\alpha_2 \leq C |E^T_{\alpha_1}| |F^\alpha_1|.
\]
The functions \( U^\varepsilon \) and \( O_0 \) are both bounded because they take values in the compact set \( \tilde{G}_1 \). In the following we make heavy use of the structure assumption 6.25 for \( A(U^\varepsilon) \) and \( A_0(U^\varepsilon) \).

As a consequence we have
\[
\partial_t A_0(\varepsilon) = \varepsilon^2 \left[ \begin{array}{cc} 0 & \partial_t A_{012}^2(\varepsilon, \tilde{w}^\varepsilon) \\ \partial_t A_{012}^2(\varepsilon, \tilde{w}^\varepsilon) & 0 \end{array} \right],
\]
\[
(A_0(U^\varepsilon)A(U^\varepsilon))_x = \varepsilon^2 B(\tilde{U})_{x},
\]
\[
A(U^\varepsilon) - A(U^\varepsilon)_m = \varepsilon^2 \left[ \begin{array}{cc} 0 & 0 \\ 0 & * \end{array} \right],
\]
\[
A_0(U^\varepsilon) - A_0(\varepsilon)_m = \varepsilon^2 \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right].
\]
Here \( B(\tilde{U})_x \in O(1) \). Recall that \( A, A_0 \in C^\infty \). So we can estimate with (6.32) the remaining terms as
\[
I^\alpha_3 \leq C |E_{\alpha}| |F^\alpha_2|,
I^\alpha_4 \leq C(\varepsilon + \varepsilon^2) |E_{\alpha}|^2 \leq C |E_{\alpha}|^2;
I^\alpha_5 \leq C |E_{\alpha}|^2, 
I^\alpha_6 \leq C \varepsilon |E_{\alpha}| |F^\alpha_1|.
\]

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By integrating equation (6.31) with respect to $x$ over $\Omega$ and using the periodicity of the initial data we have proven the assertion.

We come to the right hand side of the inequality in lemma 6.12.

**Lemma 6.13**

With

$$\Delta := \frac{\|U^m - U^\varepsilon\|}{\varepsilon^2}$$

the following inequalities hold

$$C\|E^{II}_\alpha\|\|F^\alpha_1\| \leq \frac{C_0}{\varepsilon^2} \|E^{II}\|^2 + C\varepsilon^2 \|E^{II}\|_{\alpha-1}^2$$

$$+ C\Delta^2\|E\|_{\alpha}^2 + C(1 + \Delta)(1 + \Delta^s)\|E\|_{\alpha}^2$$

(6.33)

$$C\varepsilon\|E_\alpha\|\|F^\alpha_1\| \leq C\varepsilon^{2m} + \frac{C}{\varepsilon^2} \|E^{II}\|_{\alpha-1}^2 + C\Delta^2\|E\|_{\alpha}^2$$

$$+ C(1 + \Delta)(1 + \Delta^s)\|E\|_{\alpha}^2$$

(6.34)

$$\|F^\alpha_2\| \leq C\varepsilon^{m} + C(1 + \Delta^s)\|E\|_{\alpha}.$$  

(6.35)

Here $\varepsilon < 1$ and $\|\cdot\|_{-1} := 0$.

**Proof.** Consider $\|F^\alpha_1\|$. Observe that

$$Q(U^m_\varepsilon) - Q(U^\varepsilon) - Q(U_0) = \int_0^1 \int_0^1 U(\Theta)Q_{UU}(O_0 + \eta U(\Theta)) \, d\eta \, d\Theta \cdot E,$$

with $U(\Theta) = U^m_\varepsilon - O_0 + (1 - \Theta)(U^\varepsilon - U^m_\varepsilon)$. It follows from 6.7-3) and lemma 6.5 that

$$\|Q_{UU}(O_0 + \eta U(\Theta))\| \leq C(1 + \|U(\Theta)\|_s^s).$$

With $\|U(\Theta)\|_s \leq \|U^m_\varepsilon - O_0\|_s + \|U^\varepsilon - U^m_\varepsilon\|_s$ and (6.26) it follows

$$\|U(\Theta)\|_s \leq C\varepsilon + \varepsilon^2\Delta \leq C(1 + \varepsilon^2\Delta).$$

Hence,

$$\|(Q(U^m_\varepsilon) - Q(U^\varepsilon) - Q(U_0) E)\|_{\alpha}$$

$$\leq C\|(Q(U^m_\varepsilon) - Q(U^\varepsilon) - Q(U_0) E)\|_{\alpha}$$

$$\leq C\int_0^1 \int_0^1 \|U(\Theta)\|_{\alpha} \|Q_{UU}(O_0 + \eta U(\Theta))\|_{\alpha} \, d\eta \, d\Theta \|E\|_{\alpha}$$

$$\leq C\int_0^1 \int_0^1 \|U(\Theta)\|_s \|Q_{UU}(O_0 + \eta U(\Theta))\|_s \, d\eta \, d\Theta \|E\|_{\alpha}$$

$$\leq C(\varepsilon + \varepsilon^2\Delta)(1 + \varepsilon^2\Delta^s)\|E\|_{\alpha}.$$
Now 6.7-4) and lemma 6.5 ii) yield
\[
\|F^\alpha_1\| \leq \varepsilon^{m-1}\|(Q_U(O_0)O_{m+1})_\alpha\| + \frac{1}{\varepsilon^2}\|(Q_U(O_0)E)_\alpha - Q_U(O_0)E\|_\alpha \\
+ \frac{1}{\varepsilon^2}\|(Q(U^m_\varepsilon) - Q(U^\varepsilon) - Q_U(O_0)E\|_\alpha \\
\leq C\varepsilon^{m-1} + \frac{C}{\varepsilon^2}\|E^{II}\|_{\alpha-1} + \frac{C}{\varepsilon}(1 + \varepsilon\Delta)(1 + \varepsilon^2\Delta^s)\|E\|_\alpha \\
\leq C\varepsilon^{m-1} + \frac{C}{\varepsilon^2}\|E^{II}\|_{\alpha-1} + \frac{C}{\varepsilon}\|E\|_\alpha + C\varepsilon^{2s-1}\|E\|_\alpha + C\Delta(1 + \varepsilon^2\Delta^s)\|E\|_\alpha.
\]

The remaining inequalities unfold with help of Young’s inequality A.3 as follows

\[
C\|E^{II}_\alpha\|\|F^\alpha_1\| \leq C\frac{\|E^{II}\|}{\varepsilon}\left(\varepsilon^m + \frac{1}{\varepsilon}\|E^{II}\|_{\alpha-1} + \|E\|_\alpha + \varepsilon^2s\Delta^s\|E\|\right) \\
+ C\Delta(1 + \varepsilon^2s\Delta^s)\|E\|_\alpha^2 \\
\leq \frac{C_0}{2\varepsilon^2}\|E^{II}\|^2 + C\varepsilon^{2m} + \frac{C}{\varepsilon^2}\|E^{II}\|_{\alpha-1}^2 + C\|E\|^2_\alpha + C\varepsilon^4s\Delta^2\|E\|^2_\alpha \\
+ C\Delta(1 + \varepsilon^2s\Delta^s)\|E\|^2_\alpha \\
\leq \frac{C_0}{2\varepsilon^2}\|E^{II}\|^2 + C\varepsilon^{2m} + \frac{C}{\varepsilon^2}\|E^{II}\|_{\alpha-1}^2 + C\Delta^2\|E\|^2_\alpha \\
+ C(1 + \Delta)(1 + \Delta^s)\|E\|^2_\alpha \\
C\varepsilon\|E\|\|F^\alpha_1\| \leq C\|E\|\left(\varepsilon^m + \frac{1}{\varepsilon}\|E^{II}\|_{\alpha-1} + \|E\|_\alpha + \varepsilon^2s\Delta^s\|E\|\right) \\
+ C\varepsilon\Delta(1 + \varepsilon^2s\Delta^s)\|E\|_\alpha^2 \\
\leq C\|E\|^2 + C\varepsilon^{2m} + \frac{C}{\varepsilon^2}\|E^{II}\|_{\alpha-1}^2 + C\|E\|^2_\alpha + C\varepsilon^4s\Delta^2\|E\|^2_\alpha \\
+ C\varepsilon\Delta(1 + \varepsilon^2s\Delta^s)\|E\|^2_\alpha \\
\leq C\varepsilon^{2m} + \frac{C}{\varepsilon^2}\|E^{II}\|_{\alpha-1}^2 + C\Delta^2\|E\|^2_\alpha \\
+ C(1 + \Delta)(1 + \Delta^s)\|E\|^2_\alpha.
\]

Now let us estimate \(\|F^\alpha_2\| = \sum_{\alpha=1}^{\beta} \|f^{\alpha}_3\|\) with lemma 6.5. For \(f^\alpha_3\) we have with (6.28)
\[
\|f^\alpha_3\| \leq \varepsilon^{m-1}\|(F_m)_\alpha\| \leq C\varepsilon^m.
\]

For \(f^{\alpha}_1\) we have with the special form of the matrices 6.25, the boundedness of \(\|U^m_\varepsilon\|_{s+1}\), and
\[
A(U^\varepsilon) - A(U^m_\varepsilon) = -\int_0^1 A_U(U^m_\varepsilon + \Theta(U^\varepsilon - U^m_\varepsilon)) \, d\Theta E
\]
the following relation
\[
\|f^\alpha_1\| \leq \frac{C}{\varepsilon}\|U^m_\varepsilon\|_s \|A_U(U^m_\varepsilon + \Theta(U^\varepsilon - U^m_\varepsilon))\|E\|_\alpha \leq \frac{C}{\varepsilon}\|A_U|_{s}(1 + \|U^m_\varepsilon + \Theta(U^\varepsilon - U^m_\varepsilon)\|_{s})\|E\|_\alpha \\
\leq C(1 + (\|U^m_\varepsilon\|_s + \|U^\varepsilon - U^m_\varepsilon\|_{s}))\|E\|_\alpha \leq C(1 + \Delta^s)\|E\|_\alpha.
\]
As short explanation for the latter inequality note that the term \((\|U_m\|_s + \|U^- - U_m\|_s)^s\) incorporates elements that can be bounded by \(C\Delta^{s-k}\) for some \(0 < k < s\). If \(\Delta \leq 1\) we have \(C\Delta^{s-k} \leq C\) and for \(\Delta > 1\) we infer \(C\Delta^{s-k} \leq C\Delta^s\). For \(f^{\alpha}_2\) we have

\[
\|f^{\alpha}_2\| \leq \frac{C}{\varepsilon}\|A(U)\alpha E\| \leq \frac{C}{\varepsilon}\|E\|_\alpha\|A(U)\alpha\| \leq C\|E\|_\alpha.
\]

Hence,

\[
\|F^\alpha_2\| \leq C\varepsilon^m + C(1 + \Delta^s)\|E\|_\alpha.
\]

Let us plug (6.33)-(6.35) into the inequality in lemma 6.12 to infer

\[
\frac{d}{dt} \int_\Omega e(E_\alpha) \, dx + \frac{c_0}{2\varepsilon^2} \|E^{II}_\alpha\|^2 \leq C\varepsilon^{2m} + \frac{C}{\varepsilon^2} \|E^{II}\|_{\alpha-1}^2 + C\Delta^{2s}\|E\|_\alpha^2
\]

\[
+ C(1 + \Delta)(1 + \Delta^s)\|E\|_\alpha^2
\]

\[
\leq C\varepsilon^{2m} + \frac{C}{\varepsilon^2} \|E^{II}\|_{\alpha-1}^2 + C(1 + \Delta^{2s})\|E\|_\alpha^2.
\]

With \(C^{-1}|E^\alpha|^2 \leq e(E^\alpha) \leq C|E^\alpha|^2\) integration of (6.36) from 0 to \(T\) with \(T \leq \min\{T_\varepsilon, T_m\}\) yields

\[
\|E_\alpha(T)\|^2 + \frac{1}{\varepsilon^2} \int_0^T \|E^{II}_\alpha\|^2 \, dt \leq C\varepsilon^{2m} + \frac{C}{\varepsilon^2} \int_0^T \|E^{II}\|_{\alpha-1}^2 \, dt
\]

\[
+ C(1 + \Delta^{2s}) \int_0^T \|E\|_\alpha^2 \, dt.
\]

We used \(\|E(0)\|_s = O(\varepsilon^m)\) from Assumption 6.7. By summing the latter inequality for \(\alpha \leq k, 0 \leq k \leq s\), we obtain

\[
\|E(T)\|_k^2 + \frac{1}{\varepsilon^2} \int_0^T \|E^{II}\|_k^2 \, dt \leq C\varepsilon^{2m} + \frac{C}{\varepsilon^2} \int_0^T \|E^{II}\|_{k-1}^2 \, dt
\]

\[
+ C(1 + \Delta^{2s}) \int_0^T \|E\|_k^2 \, dt
\]

(6.37)

An iteration argument on (6.37) gives

\[
\frac{1}{\varepsilon^2} \int_0^T \|E^{II}\|_k^2 \, dt \leq C\varepsilon^{2m} + C(1 + \Delta^{2s}) \int_0^T \|E\|_k^2 \, dt.
\]

(6.38)

In order to preserve the flow of the proof, the proof of the above relation can be found in the Appendix A.

The latter inequality (6.38) and (6.37) for \(k = s\) yield

\[
\|E(T)\|_s^2 \leq C\varepsilon^{2m} + (1 + \Delta^{2s}) \int_0^T \|E\|_s^2 \, dt.
\]

(6.39)
We apply Gronwall’s lemma A.2 to (6.39) and obtain
\[ \|E(T)\|_s^2 \leq C T_m \varepsilon^{2m} \exp\left(C T(1 + \Delta^{2s})\right). \] (6.40)

With \( \|E\|_s = \varepsilon^2 \Delta \), from (6.40) we have
\[ \Delta(T)^2 \leq C T_m \varepsilon^{2m-4} \exp\left(C T(1 + \Delta^{2s})\right) =: \Phi(T). \] (6.41)

It holds
\[ \Phi'(t) = C(1 + \Delta^{2s})\Phi(t) \leq C\Phi(t) + C\Phi(t)^{s+1}. \]

We apply lemma 6.6 to obtain
\[ \sup_{[0,T_m]} \Phi(t) \leq \exp(C T_m), \]
if \( m > 2 \) and \( \varepsilon \) sufficiently small, such that \( \Phi(0) = C T_m \varepsilon^{2m-4} < \delta \). Due to (6.41) there exists a constant \( c \), independent of \( \varepsilon \), such that
\[ \Delta(T) \leq c; \] (6.42)
for any \( T \in [0, \min\{T_c, T_m\}] \). The assertion follows from (6.41) with (6.42). \( \square \)
6.2 Application to the problem at hand

We finally present our main theorem of this section.

**Theorem 6.14** (Parabolic asymptotic limit of (Euler-Darcy-MS))

Let \(d = 1, s \geq 2\), the mobility constants \(M_i > 0, i = 1, \ldots, n\), \(p_i'(\rho_i) \in (0, \infty)\), \(i = 1, \ldots, n\), and \(\lambda_{ij}\) as in theorem 4.2. Consider the system (Euler-Darcy-MS-ND) in the regime (AS). Let the initial data \(U_0(\cdot; \varepsilon) \in C^1\) have vanishing zero order \(\varepsilon\) terms in the momenta and compact support. Further, let the solution \(U^\varepsilon \in H^s\) of (6.2) have vanishing zero and first order \(\varepsilon\) terms in the momenta.

For \(\varepsilon \to 0\) the solution \(U^\varepsilon\) of (6.2) converges to the solution \(O_0\) of the parabolic limit system (6.3) and it holds

\[
\|U^\varepsilon(t) - O_0(t)\|_{C^1} \leq C\varepsilon,
\]

for all \(t \in [0, T_m]\).

**Remark 6.15.**

- The condition on the derivatives of the pressure functions is restrictive but allows for instance the perfect gas law.
- We have a condition on the initial data, i.e. vanishing zero order part in the momenta, it suggests itself to demand vanishing first order part as well, to make the condition on \(U^\varepsilon\) reasonable.

**Proof.** Obviously the system (6.2) has the form (6.5). We want to apply theorem 6.11 to our system (6.2). Let us check all the assumptions.

First we consider the parabolic structural assumption (PS). From (6.23), \(p_i' > 0, i = 1, \ldots, n\), and (6.1) it is obvious that \(q(r, m) = 0 \iff m = 0\). The second condition has already been proven in section 4, see step one in proof of theorem 4.2.

Recall the structure of \(A\) and \(Q\) for the system (6.2) as mentioned in (6.22), (6.23). The structural assumption (6.25) is satisfied, because \(p_i'(\rho_i)\) are constant for all \(i = 1, \ldots, n\). Additionally the solution \(U^\varepsilon\) does not have zero and first order terms in \(\varepsilon\) for the momentum part.

Consider the stability condition. Let

\[
A_0(U) = \begin{pmatrix} A(U) & B(U) \\ B(U) & D(U) \end{pmatrix}
\]

be a symmetric matrix, where all blocks \(A, B,\) and \(D\) are diagonal. We use the abbreviation \(\text{diag}(x_1, \ldots, x_n) = \text{diag}(x_i)\). It holds

\[
A_0(U)A(U) = \begin{pmatrix} B \text{diag}(p_i'(\rho_i)) - B \text{diag} \left( \frac{m_i^2}{\rho_i^2} \right) & A + B \text{diag} \left( \frac{2m_i}{\rho_i} \right) \\ D \text{diag}(p_i'(\rho_i)) - D \text{diag} \left( \frac{m_i^2}{\rho_i^2} \right) & B + D \text{diag} \left( \frac{2m_i}{\rho_i} \right) \end{pmatrix}
\]
6.2 Application to the problem at hand

This matrix should be symmetric. Hence, it needs to hold

\[ A + B \text{ diag } \left( \frac{m_i}{\rho_i} \right) = D \text{ diag } (p'_i(\rho_i)) - D \text{ diag } \left( \frac{m^2_i}{\rho^2_i} \right). \]

With \( m_i = \varepsilon^2 \tilde{m}_i \) this can be achieved with the choice \( A = D \text{ diag } (p'_i(\rho_i)) = \text{ const.} \), \( B(\tilde{U}) = -\varepsilon^2 D \text{ diag } (\frac{m_i}{\rho_i}) \), and \( D > 0 \). The eigenvalues of \( A_0(U) \) in this case are positive due to the diagonal structure of the blocks and the fact that

\[ \text{eig} \left( \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) = \left\{ \frac{1}{2} \left( a + d + \sqrt{(a - d)^2 + 4b^2} \right), \frac{1}{2} \left( a + d - \sqrt{(a - d)^2 + 4b^2} \right) \right\}. \]

For \( a = dp'_i(\rho), b = -\varepsilon^2 d \tilde{m}_i/\rho, \) and \( d > 0 \) the eigenvalues equal

\[ \frac{d(1 + p'_i(\rho))}{2} \pm \frac{1}{2} d \sqrt{(p'_i(\rho) - 1)^2 + 4\varepsilon^4 \tilde{m}^2/\rho^2} > 0, \]

for sufficiently small \( \varepsilon \). So

\[ A_0(U) = \begin{pmatrix} D \text{ diag } (p'_i(\rho_1), \ldots, p'_n(\rho_n)) & -\varepsilon^2 D \text{ diag } \left( \frac{\tilde{m}_1}{\rho_1}, \ldots, \frac{\tilde{m}_n}{\rho_n} \right) \\ -\varepsilon^2 D \text{ diag } \left( \frac{\tilde{m}_1}{\rho_1}, \ldots, \frac{\tilde{m}_n}{\rho_n} \right) & D \end{pmatrix} \]

is symmetric positive definite and the first part of the stability condition holds. Note that \( A_0(O_0) = A_0(\rho_0, 0) \) is block diagonal as mentioned in the proof of theorem 6.11.

Let us consider the second part of the stability condition. With \( A_0(U) \) as derived above and the structure (4.8) of \( Q_U \), we have for \( \bar{U} \) with \( Q(\bar{U}) = 0 \)

\[ A_0(\bar{U}) Q_U(\bar{U}) = \begin{pmatrix} 0 & 0 \\ 0 & D q_w(\bar{U}) \end{pmatrix}, \]

because \( B(\bar{U}) = 0 \), due to \( m_i = 0, \) \( i = 1, \ldots, n \). As proven in part i) of the proof of theorem 4.2, we have \( q_w(\bar{U}) < 0 \) and with \( D > 0 \) trivially existence of a matrix \( S(\bar{U}) > 0 \) such that the second part of the stability condition is fulfilled.

Finally, we need to verify the remaining parts of assumption 6.7. The condition 1) is obviously true. 2) depends only on the initial data. With the lemma 6.2 and the existence theory for parabolic systems 3) follows immediately. As mentioned in remark 6.8 the items 4) and 5) are fulfilled.

We verified all assumptions and are able to apply theorem 6.11. Therefore it holds for the approximation \( U^m_{\varepsilon} \)

\[ \| U^1_{\varepsilon}(t) - U^\varepsilon(t) \|_s \leq K\varepsilon, \]

for \( t \in [0, T_m] \). With the Sobolev embedding theorem A.1, we infer

\[ \| U^\varepsilon - O_0 \|_{C^1} \leq \| U^\varepsilon - U^1_{\varepsilon} \|_{C^1} + \varepsilon \| O_1 \|_{C^1} \leq K\varepsilon + \varepsilon \| O_1 \|_{C^1} \leq C\varepsilon. \]

This finishes the proof.


7 Conclusions

In this thesis we derived a set of partial differential equations which describe the dynamics of an inviscid, isothermal, compressible fluid mixture in a porous medium. We did this in a way, that the solutions to the system automatically satisfy an entropy inequality. The derived model fulfils the second law of thermodynamics. The character of the system was then analysed.

Using the work of Yong [22] for more general hyperbolic systems, we were able to prove global-in-time existence of smooth solutions in one space dimension. Our theorem incorporates the special case of one component, for which the assertion was already known. An important condition of this theorem is the positivity of the mobility constants. That means that the porous medium needs to be present in order to have smooth solutions. This fact corresponds with our expectation. It is known that discontinuities occur when the porous medium is missing. The numerical experiments approved our theoretical result. We used an explicit finite volume scheme with Roe flux to perform the simulations. In case of one component and in the case of multiple components we observed that vanishing mobility constants lead to discontinuities in the solution which did not occur for positive mobility constants. It was surprising that even non-smooth initial data lead to solutions which were smooth after small time. This shows how drastic the diffusive effects due to the porous medium and the binary interactions are. Our theoretical result does not capture this observation. For future work a proof for the case of non-smooth initial data should be considered. It is obvious to suspect that solvers for general hyperbolic system can not work in every regime for our system, because as discussed it changes in certain regimes its character from hyperbolic to parabolic. A proper numerical algorithm which can deal with this fact is desirable. In [7] Degond et al. suggested so called asymptotic-preserving schemes for this purpose. The ideas might be transferred to our system. We leave this task for future work.

A large part of this thesis deals with the question of the existence of a parabolic limit system. We were able to prove convergence to a limit system in one space dimension with help of techniques from [21] and [14]. The special case of one component for the derived limit system matches the one from existence results [13]. In order to perform our proof, we had to make some strict assumptions. Namely, one space dimension, special form of the initial data (basically small velocities), a certain order for the velocities in the unknown solution, and constant derivatives of the pressure functions. The restriction of one space dimension can be quite easily expanded to several space dimensions. As mentioned in the corresponding section, the result can be generalized to generic initial data with help of another expansion and a matching process. The restriction on the pressure function and the assumption on the solution are not that easy to overcome. This issue requires further investigation. Overall we were able to transfer the presented results for the one component case under certain circumstances to the multicomponent case.
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References


References


A Appendix

**Theorem A.1** (Sobolev embedding theorem [10])

Let $U$ be a bounded, open subset of $\mathbb{R}^n$ and suppose $\partial U$ is $C^1$. For $p \in [1, \infty]$ assume $u \in W^{k,p}(U)$. If $k > \frac{n}{p}$, $k \in \mathbb{N}$ then $u \in C^{k-[n/p]-1,\gamma}(\bar{U})$, where

$$
\gamma = \begin{cases}
\left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \notin \mathbb{N} \\
\text{any positive number in } (0,1), & \text{if } \frac{n}{p} \in \mathbb{N}.
\end{cases}
$$

In addition, there exists $C > 0$ such that

$$
\|u\|_{C^{k-[n/p]-1,\gamma}(\bar{U})} \leq C \|u\|_{W^{k,p}(U)}.
$$

**Lemma A.2** (Gronwall’s inequality [1])

Let $\Phi(\cdot)$ be a continuous function on $[0, T]$, which satisfies for all $t \in [0, T]$ the inequality

$$
\Phi(t) \leq \int_0^t \alpha(s) \Phi(s) \, ds + C,
$$

where $\alpha$ is a nonnegative, continuous function on $[0, T]$ and $C$ a constant. Then

$$
\Phi(t) \leq C \exp \left( \int_0^t \alpha(s) \, ds \right)
$$

for all $0 \leq t \leq T$.

**Lemma A.3** (Young’s inequality [10])

Let $a, b \in (0, \infty)$, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
$$
**Proof of relation (6.38)**

Let $k = 0$. With $\| \cdot \|_{-1} = 0$ and inequality (6.37) we have

$$\frac{1}{\varepsilon^2} \int_0^T \| E^{II} \|^2 \leq \| E(T) \|^2 + \frac{1}{\varepsilon^2} \int_0^T \| E^{II} \|^2 \leq CT \varepsilon^{2m} + C(1 + \Delta^2 s) \int_0^T \| E \|^2 \, dt.$$ 

So (6.38) is true for $k = 0$. We assume the assertion to be true for some $k \in \mathbb{N}$. For $k + 1$ it holds

$$\frac{1}{\varepsilon^2} \int_0^T \| E^{II} \|^2 \leq \| E(T) \|^2 + \frac{1}{\varepsilon^2} \int_0^T \| E^{II} \|^2 \leq CT \varepsilon^{2m} + \frac{C}{\varepsilon^2} \int_0^T \| E^{II} \|^2 \, dt + C(1 + \Delta^2 s) \int_0^T \| E \|^2 \, dt \leq CT \varepsilon^{2m} + C(1 + \Delta^2 s) \int_0^T (\| E \|^2_k + \| E \|^2_{k+1}) \, dt \leq CT \varepsilon^{2m} + C(1 + \Delta^2 s) \int_0^T \| E \|^2_{k+1} \, dt.$$ 

Hence (6.38) holds for all $k \in \mathbb{N}$. □
B Software

For all computations we used MATLAB®.

You can find several matlab files on the CD attached to this thesis amongst the movies to the examples in section 5. We explain here shortly how to use them in order to perform your own simulations. There are some auxiliary files which should be left untouched. Their function is briefly stated in the commentary of the corresponding file. All files have a short help, which can be called by using help 'functionname' in the command window. In the following we introduce the important files and their functionality. At first one should modify the files dataonecomp.m, datatwocomp.m depending whether a simulation for one or two components is desired. In these files the important parameter are specified. Namely,

- the spatial stepsize deltaX,
- the end time T for the time interval [0,T] where the simulation runs,
- the initial data uZero,
- the the type of boundary conditions. One can choose between 'periodic' and 'neumann',
- The mobility constants M1,
- In case of two components the Maxwell–Stefan coefficients lambda,
- The constants c for the pressure functions \( p_i(\rho_i) = c_i \rho_i \).

After these parameters are fixed, the corresponding script one_comp.m or two_comp.m should be run. It computes the solution uROE of the problem with the specified parameters. In order to plot the solution uROE, the total momentum \( p \), and the entropy \( E \) at a specific point in time, one needs to use the functions plotonecomp.m, plottwocomp.m. To get a movie of the whole simulation the functions makevidonecomp.m, makevidtwocomp.m are needed. They take amongst other things a string str as input. The movie is saved as str.avi in the MPEG-4 format in the same folder.
C Zusammenfassung


C Zusammenfassung


Erklärung


Stuttgart, den 20.10.2015
Published Preprints

http://www.nupus.uni-stuttgart.de

2007/1  Cao, Y. / Eikemo, B. / Helmig, R.: Fractional flow formulation for two-phase flow in porous media

2007/2  Korteland, S.-A., The average equilibrium capillary pressure-saturation relationship In two-phase flow in porous media

2008/1  Helmig, R. / Weiss, A. / Wohlmuth, B.: Variational inequalities for modeling flow in heterogeneous porous media with entry pressure

2008/2  Cao, Y. / Helmig, R. / Wohlmuth, B.: Convergence study and comparison of the multipoint flux approximation L-method

2008/3  van Duijn, C.J. / Pop, I.S. / Niessner, J. / Hassanizadeh, S.M.: Philip’s redistribution problem revisited: the role of fluid-fluid interfacial areas


2008/6  Cao, Y. / Helmig, R. / Wohlmuth, B.: Geometrical interpretation of the multipoint flux approximation L-method

2008/7  Vervoort, R.W. / van der Zee, S.E.A.T.M.: Simulating the effect of capillary flux on the soil water balance in a stochastic ecohydrological framework


2008/10 Wolff, M.: Comparison of mathematical and numerical models for two-phase flow in porous media


2008/12 Cao, Y. / Helmig, R. / Wohlmuth, B.: Convergence of the multipoint flux approximation L-method for homogeneous media on uniform grids

2008/13 Ochs, S.O.: Development of a multiphase multicomponent model for PEMFC

2008/14 Walter, L.: Towards a model concept for coupling porous gas diffusion layer and gas distributor in PEM fuel cells

2009/1  Hægland, H. / Assteerawatt, A. / Helmig, R. / Dahle, H.K.: Streamline approach for a discrete fracture-matrix system

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